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Étude mathématiques et simulations numériques de modèles de gaines bi-cinétiques.

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Mots clés : vlasov-poisson system, vlasov-ampère system, non linear poisson equation, debye sheath, parallel magnetic field, bohm criterion, floating potential

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À mes parents,

On mesure l'intelligence d'un individu à
la quantité d'incertitudes qu'il est
capable de supporter.

Emmanuel Kant

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Étude mathématiques et simulations numériques de modèles de gaines bi-cinétiques.

Abstract

Les résultats présentés dans cette thèse portent sur la construction et la simulation numérique de modèles théoriques de plasmas en présence d'une paroi absorbante. Ces modèles se basent sur des systèmes de Vlasov-Poisson ou Vlasov-Ampère à deux espèces en présence de conditions limites. Les solutions stationnaires recherchées vérifient l'équilibre des flux de charges dans la direction perpendiculaire à la paroi. Cette propriété s'appelle l'ambipolarité. A travers l'étude d'une équation de Poisson non linéaire, on montre le caractère bien posé d'un système de Vlasov-Poisson stationnaire 1d-1v pour lequel on détermine des distributions de particules entrantes et un potential au mur qui induisent l'ambipolarité et une densité de charge positive. On donne également une estimation de la taille de la couche limite au mur. Ces résultats sont illustrés numériquement. On prouve ensuite la stabilité linéaire des solutions stationnaires électroniques pour un modèle de Vlasov-Ampère instationnaire. Enfin, on étudie un modèle de Vlasov-Poisson stationnaire 1d-3v en présence d'un champ magnétique constant et parallèle à la paroi. On détermine les distributions de particules entrantes et un potentiel au mur qui induisent l'ambipolarité. On étudie une équation de Poisson non linéaire associée au modèle à l'aide d'une fonctionnelle non linéaire d'énergie qui admet des minimiseurs. On établit des bornes de paramètres à l'intérieur desquelles notre modèle s'applique et on propose une interprétation des résultats.

Keywords: système de vlasov-poisson, système de vlasov-ampère, equation de poisson non linéaire, gaine de debye, champ magnétique parallèle, critère de bohm, potential flottant

Résumé

This thesis focuses on the construction and the numerical simulation theoretical models of plasmas in interaction with an absorbing wall. These models are based on two species Vlasov-Poisson or Vlasov-Ampère systems in the presence of boundary conditions. The expected stationary solutions must verify the balance of the flux of charges in the orthogonal direction to the wall. This feature is called the ambipolarity. Through the study of a non linear Poisson equation, we prove the well-posedness of 1d-1v stationary Vlasov-Poisson system, for which we determine incoming particles distributions and a wall potential that induces the ambipolarity as well as a non negative charge density hold. We also give a quantitative estimates of the thickness of the boundary layer that develops at the wall. These results are illustrated numerically. We prove the linear stability of the electronic stationary solution for a non-stationary Vlasov-Ampère system. Finally, we study a 1d-3v stationary Vlasov-Poisson system in the presence of a constant and parallel to the wall magnetic field . We determine incoming particles distributions and a wall potential so that the ambipolarity holds. We study a non linear Poisson equation through a non linear functional energy that admits minimizers. We established some bounds on the numerical parameters inside which, our model is relevant and we propose an interpretation of the results.

Mots clés : vlasov-poisson system, vlasov-ampère system, non linear poisson equation, debye sheath, parallel magnetic field, bohm criterion, floating potential

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Chapter 1

Introduction générale

1.1 Contexte

Cette thèse, réalisée au Laboratoire Jacques Louis Lions, s'intéresse aux problématiques liées à la fusion par confinement magnétique, et plus généralement dans le contexte du projet international ITER [34] (International Thermonuclear Experiment Reactor), qui vise à démontrer la faisabilité scientifique et technologique de l'énergie de fusion. La fusion nucléaire est une réaction au cours de laquelle deux atomes légers ayant vaincu la force de répulsion Coulombienne, s'assemblent pour former un atome plus lourd. Au cours de cette réaction, et du fait de la différence de masse entre les sous produits et les réactifs, il y a production d'énergie. La réaction de fusion privilégiée pour ITER, est celle de deux isotopes de l'hydrogène, le deutérium et le tritium

$$ {}_1^2H + {}_1^3H \rightarrow {}_2^4H_e + {}_0^1n, $$

dont le résultat est la production d'un atome d'hélium non radioactif, l'émission d'un neutron et le dégagement d'une énergie $E = 17.6MeV$ qui est environ 4.6 fois l'énergie de fission de l'uranium 235. Il s'agit d'une source d'énergie plus efficace (en un certain sens) que la fission et moins polluante. Néanmoins, cette réaction est difficile à réaliser car les forces nucléaires qui lient les nucléons n'agissent qu'à très faible distance tandis que la force Coulombienne crée une barrière répulsive qui empêche les noyaux des atomes, chargés positivement, de s'approcher assez près les uns des autres. Pour franchir cette barrière, les noyaux doivent donc posséder une énergie cinétique très grande.

Lorsqu'un gaz est porté à une température suffisamment élevée (au delà de 10^5 K), l'énergie moyenne d'agitation thermique du gaz est suffisante pour que les atomes s'ionisent par collision, libérant des électrons dans le milieu qui devient ainsi conducteur. On obtient alors un mélange globalement neutre de particules chargées constitué d'ions et d'électrons appelé plasma. Ce terme a été introduit pour la première fois par I.Langmuir en 1928. L'élévation de la température n'est pas le seul moyen d'obtenir un plasma. En fait, il suffit de provoquer un mécanisme d'ionisation à faible densité. C'est d'ailleurs sur ce principe que la plupart des plasmas de laboratoires sont fabriqués.

Dans le projet ITER, une chambre à vide est alimentée d'un gaz de Deuterium-Tritium qui est ensuite chauffé à très haute température se transformant ainsi en plasma. L'étape suivante consiste alors à maintenir une densité et une température de plasma suffisamment élevée pour que les réactions de fusions se produisent. L'un des inconvénients est que les parois de la chambre à vide constituent un véritable puits à particules, et que par ailleurs leur détérioration (par érosion) occasionne un risque de contamination du plasma. Aussi, afin d'éviter que des particules de poussières, par effet de transport, ne viennent se recombiner au coeur du plasma, de puissants champs magnétiques sont utilisés pour confiner le plasma à l'intérieur la chambre. En pratique, il existe toujours une fraction de particules qui vient heurter les parois, c'est pourquoi dans le tokamak ITER, on trouve au niveau du plancher, une structure appelée diverteur (il s'agit de la structure orange sur la figure 1.1) dont la fonction est d'assurer l'extraction des impuretés et des effluents gazeux. Cette structure est composée de trois éléments cibles qui font face au plasma, situés à proximité des premières lignes de champs magnétiques ouvertes (dans le plan poloïdal), là où les particules les plus énergétiques viennent percuter les composants. Ces cibles doivent pouvoir supporter des charges thermiques surfaciques très élevées (de l'ordre de 10 à 20 $MW.m^{-2}$). Le matériau qui a été choisi pour la fabrication des diverteurs est le tungstène, un métal qui présente une résistance à l'usure importante.

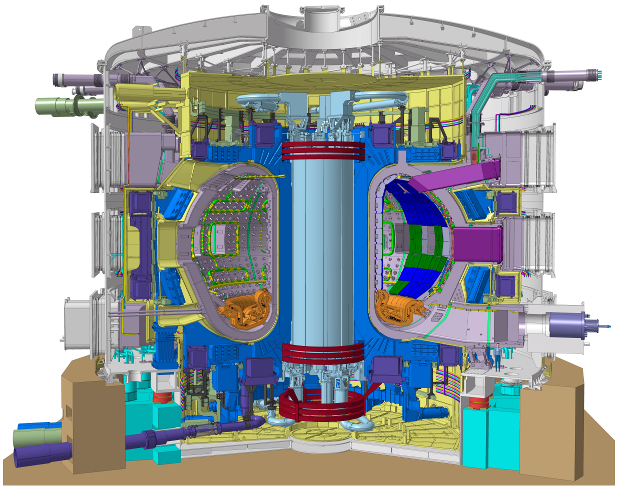


Figure 1.1 – Tokamak ITER, source: www.iter.org

L'interaction entre un plasma et une paroi est un sujet de recherche important en physique des plasmas [12, 57, 18, 44, 64]. C'est un sujet délicat qui canalise encore les recherches, les études scientifiques visent à comprendre et à mieux modéliser cette interaction. Ces études présentent un intérêt pour la fusion: par exemple, avoir un modèle qui permet de quantifier les flux des particules percutant les diverteurs permet de mieux estimer leur durée de vie, mais aussi d'évaluer l'impact des diverteurs sur le plasma [44]. Il est donc nécessaire de modéliser, d'étudier et de simuler numériquement les phénomènes physiques mis en jeu dans cette interaction. A l'heure actuelle, il existe beaucoup de modèles dont l'objectif est justement de mieux prendre en compte cette interaction dans l'étude des plasmas de fusion. Néanmoins, le domaine de validité de ces modèles est souvent une question ouverte, et leur simulation numérique est souvent coûteuse et ne permet pas toujours de donner des réponses entièrement satisfaisantes. Dans cette thèse nous proposons d'étudier certains systèmes d'équations aux dérivées partielles susceptibles de modéliser une interaction plasma-paroi et d'en simuler numériquement les solutions. Le but des modèles étudiés ici est d'enrichir les outils théoriques existant pour l'étude de ces intéractions et de compléter la compréhension et la validation numérique des simulations existantes.

1.2 Définition et échelles caractéristiques dans les plasmas

Un plasma est un gaz de particules chargées qui est globalement neutre. Une des particularités des plasmas est l'existence d'effets collectifs qui prévalent sur les interactions binaires entre particules. Un exemple d'effet collectif est l'écrantage de Debye qui traduit le fait que la présence de nombreuses particules chargées a pour effet de limiter fortement le rayon d'action des forces électrostatiques dues à un déséquilibre local de charge. Il est utile d'introduire certaines échelles physiques caractéristiques.

L'échelle de temps est caractérisée par la fréquence plasma qui donne un ordre de grandeur du temps qu'un plasma, initialement dans une position d'équilibre, met pour retrouver cette position si on le perturbe légèrement. Plus précisément, si dans un plasma à l'équilibre, un groupe d'électrons est déplacé dans une direction, alors les ions (beaucoup plus lourds) vont exercer par l'intermédiaire des forces de Coulomb une force de direction opposée au déplacement de façon à rétablir leurs positions d'équilibres. Les électrons oscillent alors autour de cette position et la fréquence de leurs oscillations est donnée par la fréquence plasma

$$\omega_p = \sqrt{\frac{n_0 q^2}{\varepsilon_0 m_e}},$$

où n_0 est la densité moyenne des électrons, q la charge élementaire, m_e la masse d'un électron et ε_0 la permittivité du vide.

L'échelle d'espace est caractérisée par la longueur de Debye qui donne l'ordre de grandeur de la distance au-delà de laquelle la force Coulombienne induite par une particule est écrantée par le champ électrostatique de l'ensemble des particules. La longueur de Debye s'exprime en fonction de la température et de la densité des espèces de particules

$$\frac{1}{\lambda_D^2} = \sum_s \frac{n_{0,s} q_s^2}{\varepsilon_0 k_B T_s},$$

où s désigne l'espèce des particules, $n_{0,s}$ la densité, T_s la température et k_B la constante de Boltzmann. Dans un plasma isolé, à cause de l'effet d'écrantage de Debye, les effets de séparation de charges sont localisés et ne peuvent s'étendre sur des distances supérieures à quelques longueurs de Debye. Un plasma a donc une tendance naturelle à neutraliser les décharges électriques. Lorsqu'on observe un plasma, il convient de préciser l'échelle d'espace considérée, car en vertu de l'écrantage de Debye, nous pouvons distinguer deux types de régions dans un plasma:

- Les régions dont l'étendue est grande par rapport à la longueur de Debye. Elles peuvent être considérées comme “quasi-neutre” au sens où la charge y est nulle presque partout.
- Les régions de taille comparable à la longueur de Debye. On peut observer des déséquilibres de charge.

Il existe un autre paramètre physique important qui permet d'évaluer l'importance des interactions binaires entre particules (collisions). Il s'agit du libre parcours moyen, c'est-à-dire la distance moyenne parcourue par une particule entre deux collisions successives. Elle s'exprime pour une espèce de particule par

$$\lambda_{mfp}^s = \frac{k_B T_s}{\sqrt{2}\pi d_s^2 p},$$

où T_s désigne la température, p est la pression du milieu et d_s le diamètre de l'espèce des particules. Pour modéliser un plasma, il est donc nécessaire de préciser les régions de plasma considérées et les effets que l'on souhaite prendre en compte. Dans cette thèse, les plasmas que nous considérons sont électrostatiques et non collisionnels: on néglige les effets du champ magnétique autoconsistant et on fait l'hypothèse que le libre parcours moyen des espèces modélisées est très grand devant le diamètre du domaine d'étude. Enfin, les plasmas sont supposés en contact avec une paroi (ou un mur).

1.3 La gaine de Debye et le critère de Bohm

1.3.1 Une description qualitative

Pour un plasma au repos l'un des effets importants de la présence d'une paroi solide absorbante, en l'absence de forces extérieures, est l'effet de gaine. Nous adoptons la déscription de Rax [55]: "une gaine est une structure non-neutre, à la frontière du plasma, qui permet de préserver la quasi-neutralité à l'intérieur du plasma en régulant les flux à sa périphérie." L'une des causes de la gaine est la différence de mobilité entre ions et électrons. En effet, les électrons étant beaucoup plus légers que les ions, ils sont beaucoup plus mobiles et le flux d'électrons allant de l'intérieur du plasma vers la paroi est plus grand que celui des ions. Les électrons sont donc absorbés par la paroi plus rapidement que les ions et une région positivement chargée se forme proche de la paroi. Seulement, en raison du phénomène d'écrantage décrit plus haut, la taille de cette région se limite à quelques longueurs de Debye. Aussi, pour que le plasma reste au repos et ne se vide complètement de ses électrons, le champ électrique induit doit ré-équilibrer les flux de charges au mur: il accélère les ions (lourds) et ralentit les électrons (légers).

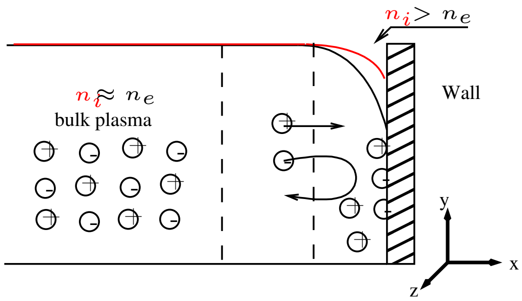


Figure 1.2 – Schéma d'un plasma électrostatique fait d'ions et d'électrons en contact avec un mur métallique. Proches du mur, les ions sont accélérés et les électrons sont ralentis par le champ éléctrique de façon à équilibrer les flux d'ions et d'électrons au mur. La gaine correspond à la region où la charge est strictement positive ($n_i > n_e$) et sa longueur est de l'ordre de quelques longueurs de Debye.

1.3.2 Un modèle simplifié de gaine et le rôle du critère de Bohm

Nous présentons maintenant un modèle simplifié de gaine qui n'est pas celui étudié dans la thèse mais qui permet de décrire plus en détails le comportement de le gaine et d'introduire un critère important en physique: le critère de Bohm. Pour cela, nous adoptons une présentation similaire à Chen [18] et considérons un plasma semi-infini uni-dimensionnel à l'état de repos contenu dans l'intervalle $(-\infty, L]$ où $L > 0$ désigne la longueur caractéristique de la région du plasma que nous allons modéliser. La variable d'espace est notée $x \in (-\infty, L]$. On suppose que dans le coeur du plasma ($x < 0$), il existe une source ionisante qui fait entrer tous les ions en $x = 0$ avec une vitesse $u_0 > 0$. Nous notons n_e la densité des électrons dans le plasma, et n_i celle des ions. Le plasma est supposé neutre en $x = 0$, i.e, $n_i(0) = n_e(0) =: n_0 > 0$. La vitesse des ions en $x \in [0, L]$ est notée $u_i(x)$. D'autre part, les observations du phénomène physique de gaine nous conduisent à faire l'hypothèse suivante:

1. Le potentiel électrostatique dans la gaine $\phi : [0, L] \to \mathbb{R}$ décroît et $\phi(0) = 0$.

Le principe de conservation de l'énergie donne alors que pour tout $x \in [0, L)$,

$$\frac{m_i u_i(x)^2}{2} + q\phi(x) = \frac{m_i u_0^2}{2},$$

d'où l'on déduit que $u_i(x) = \sqrt{u_0^2 - (2q\phi(x)/m_i)}$. L'équation de continuité donne également que pour tout $x \in [0, L)$,

$$n_i(x)u_i(x) = n_0 u_0$$

donc

$$n_i(x) = \frac{n_0}{\sqrt{1 - \frac{2q\phi(x)}{u_0^2 m_i}}}.$$

Par ailleurs, nous supposons que les électrons sont Boltzmanniens, c'est-à-dire que la densité des électrons est donnée pour $x \in [0, L]$ par

$$n_e(x) = n_0 e^{\frac{q\phi(x)}{k_B T_e}}. \tag{1.1}$$

Enfin la loi de Gauss stipule que ϕ satisfait l'équation de Poisson

$$-\phi''(x) = \frac{q}{\varepsilon_0}(n_i - n_e)(x) = \frac{qn_0}{\varepsilon_0}\left(\frac{1}{\sqrt{1 - \frac{2q\phi(x)}{u_0^2 m_i}}} - e^{\frac{q\phi(x)}{k_B T_e}}\right) \text{ pour tout } x \in (0, L). \tag{1.2}$$

Cette équation de Poisson non linéaire est souvent appelé "the planar sheath equation"(sheath signifie gaine). D'un point de vue mathématique, il s'agit d'une variante de l'équation de Liouvillle [36] ou encore d'une généralisation de l'équation de Poisson-Boltzmann [15], c'est le terme non linéaire sur les ions qui enrichit l'équation. Dans [18, 57, 59] les auteurs approchent les solutions de cette équation en linéarisant le terme non linéaire. Le caractère bien posé de cette équation (dans le cas où la densité des ions est donnée) a déjà été étudié dans [28, 15]. Le cas où les densités dependent toutes les deux du potentiel ϕ a été traitée par Ambroso et al dans [3]. Le terme non linéaire dérive d'un potentiel abstrait de type Sagdeev [58]. Nous proposons pour ce modèle simplifié, d'introduire le critère de Bohm et de l'identifier comme étant une condition nécessaire et suffisante pour q'une solution décroissante de (1.2) ait la propriété que la charge $n_i - n_e$ soit positive dans le domaine. Voici donc un premier résultat simple que nous allons établir.

Proposition 1.3.1. *Soit ϕ une solution décroissante de l'équation de Poisson (1.2) et vérifiant $\phi(0) = 0$. Alors on a*

$$\textit{pour tout } x \in [0, L] \quad n_i(x) \geqslant n_e(x) \textit{ si et seulement si } u_0 \geqslant \sqrt{\frac{k_B T_e}{m_i}}.$$

L'inégalité $u_0 \geqslant \sqrt{\frac{k_B T_e}{m_i}}$ est le critère de Bohm tel qu'il a été obtenu pour la première fois par D. Bohm dans [12], elle exprime le fait que dans une gaine la vitesse moyenne des ions est plus grande que la vitesse acoustique ionique. Son rôle est d'assurer la positivité de la charge dans le domaine.

Pour simplifier l'analyse qui va suivre, on effectue les changements de variables

$$\chi = -\frac{q\phi}{k_B T_e}, \quad \xi = \frac{x}{\lambda_D} = x\left(\frac{n_0 q^2}{\varepsilon_0 k_B T_e}\right)^{\frac{1}{2}}, \quad \mathcal{M} = \frac{u_0}{\sqrt{k_B T_e/m_i}}.$$

L'équation de Poisson devient alors

$$\chi''(\xi) = \frac{1}{\sqrt{1+\frac{2\chi(\xi)}{\mathcal{M}^2}}} - e^{-\chi(\xi)} \text{ pour tout } \xi \in (0, \frac{L}{\lambda_D}).$$

Par ailleurs comme χ prend ses valeurs dans $[0, +\infty)$, nous considérons alors la fonction f de la variable réelle $t \in [0, +\infty)$ définie par

$$f(t) = \frac{1}{\sqrt{1+\frac{2t}{\mathcal{M}^2}}} - e^{-t} \tag{1.3}$$

On a alors le résultat suivant qui permet de prouver la proposition 1.3.1 : *la fonction f est positive sur* $[0, +\infty)$ *si et seulement si* $\mathcal{M}^2 \geqslant 1$. La preuve de ce résultat est donnée ci-dessous.

Preuve. On étudie le signe de la fonction f. Pour tout réel t on a

$$f(t) = e^{-t}\left(\frac{e^t}{\sqrt{1+\frac{2t}{\mathcal{M}^2}}} - 1\right).$$

Soit h la fonction définie pour tout réel $t \in [0, +\infty)$ par $h(t) = \frac{e^t}{\sqrt{1+\frac{2t}{\mathcal{M}^2}}} - 1$. Cette fonction est dérivable sur $[0, +\infty)$ et sa dérivée est donnée pour tout réel $t \in [0, +\infty)$ par

$$h'(t) = \frac{e^t(\mathcal{M}^2 - 1 + 2t)}{\mathcal{M}^2(1+\frac{2t}{\mathcal{M}^2})^{\frac{3}{2}}}.$$

On en déduit que $h'(t) \geqslant 0$ si et seulement si $t \geqslant \frac{1-\mathcal{M}^2}{2}$.

Condition nécessaire. On raisonne par contraposition, on suppose donc que $\mathcal{M}^2 < 1$. On déduit alors du signe de h' que h est strictement décroissante sur $[0, \frac{1-\mathcal{M}^2}{2}]$ et donc $\mathcal{M}e^{\frac{1-\mathcal{M}^2}{2}} - 1 \leqslant h(t) < 0$ pour tout $t \in (0, \frac{1-\mathcal{M}^2}{2}]$. Donc f n'est pas positive sur $[0, +\infty)$. Condition suffisante. On suppose $\mathcal{M}^2 \geqslant 1$. On en déduit alors que h' est positive sur $[0, +\infty)$ donc h est croissante et $h(t) \geqslant h(0) = 0$ pour tout $t \in [0, +\infty)$. On en déduit finalement que f est positive sur $[0, +\infty)$. □

1.3.3 Les difficultés et les questions liées au modèle

L'analyse du modèle simplifié montre que pour des solutions décroissantes (si elles existent) de l'équation de Poisson, l'inégalité de Bohm est à la fois nécessaire et suffisante pour assurer une densité de charge positive dans le domaine $[0, L]$. Les solutions vérifient donc dans ce cas deux des trois propriétés fondamentales d'une gaine, à savoir, un potentiel décroissant et une charge positive. La troisième propriété qui n'est pas assurée a priori, est l'ambipolarité des flux qui traduit le fait que les flux d'ions et d'électrons à la paroi ($x = 1$) sont égaux. Pour satisfaire cette dernière, l'idée est d'ajouter une condition limite supplémentaire sur le potentiel et d'en déterminer la valeur. Il existe alors au moins deux façon de procéder :

- comme c'est le cas dans [18] ou [57], on peut supposer que le point $x = 0$ est situé dans le coeur du plasma et que le champ électrique y est nul donc $\phi'(0) = 0$.
- ou bien comme dans [44], on peut supposer que la différence de potentiel entre l'entrée du domaine en $x = 0$ et le mur en $x = L$ est donnée par le flux de charges au mur.

La première approche n'est pas entièrement satisfaisante, car il n'est pas évident d'assurer a priori l'ambipolarité des flux. La seconde approche est celle qu'on adoptera dans cette thèse, elle consiste à determiner la valeur du potential au mur que l'on notera $\phi_w < 0$ aussi appelé potentiel flottant et une solution ϕ décroissante de (1.2) tel que $\phi(0) = 0$, $\phi(L) = \phi_w$ et tel que les flux de charges s'équilibrent au mur. Cette approche est difficile à mettre en oeuvre dans le cadre du modèle de gaine simplifié que nous avons présenté. Cela est dû au choix de modélisation que nous avons adopté et au fait que la notion de courant électronique n'est pas définie dans ce modèle. De plus, l'approximation de Boltzmann (1.1) tel que nous l'avons adopté, ne permet pas non plus de définir de façon fermée la vitesse moyenne des électrons. Enfin, remarquons que le critère de Bohm $u_0 \geqslant \sqrt{\frac{k_B T_e}{m_i}}$ ne donne aucune information sur la distribution des vitesses ioniques. Il s'agit d'une inégalité sur la vitesse moyenne des ions que nous avons obtenu grace à une modélisation macroscopique du plasma. Voici quelques questions liées à la physique de la gaine électrostatique, que nous proposons de traiter dans cette thèse:

- Comment représenter, à l'aide d'un modèle simple, la distribution des vitesses ioniques dans une gaine (voir section 2.3.1) ?
- Si les électrons sont Maxwelliens dans le coeur du plasma, sont-ils exactement Boltzmanniens dans une gaine (voir section 2.3.2) ?
- Peut-on par une formule exacte determiner (a priori) le potentiel au mur (voir 2.4) ?
- La gaine est-elle stable (voir chapitre 3) ?

1.4 La pré-gaine magnétique et le critère de Bohm-Chodura

1.4.1 Une description qualitative

Lorsqu'un plasma est magnétisé à l'aide d'un champ magnétique (externe) constant, au repos, en plus de la gaine de Debye, une autre région caractéristique précédant la gaine fait son apparition. C'est R. Chodura dans [19] qui est à l'origine de ce constat. Plus précisément, il considère une géométrie dans laquelle le champ électrique est normal à la paroi, et le champ magnétique constant fait un angle $\theta \in [0, \frac{\pi}{2})$ avec la normale à la paroi (voir figure 1.3). Il observe alors, à l'aide de simulations numériques, qu'une zone dite de pré-gaine magnétique vient faire la transition entre le coeur du plasma ($x \leqslant 0$) quasi-neutre et la gaine (proche de $x = L$). Il explique que la pré-gaine magnétique est quasi-neutre et que sa taille est de l'ordre de quelques rayon de Larmor d'ions, c'est-à-dire de l'ordre du rayon de gyration des ions autour du champ magnétique. Contrairement à la gaine de Debye où le champ électrique est dominant, dans la prégaine les deux forces, électriques et magnétiques, sont en compétition. Le champ magnétique oriente les ions parallèlement à son axe, alors que le champ électrique accélère les ions perpendiculairement à la paroi afin qu'à l'entrée de la gaine, leur vitesse moyenne perpendiculaire satisfasse le critère de Bohm. Il montre également que le potentiel électrique dans la prégaine magnétique peut avoir des oscillations mais est décroissant à l'approche de la paroi. Chodura représente également la vitesse moyenne perpendiculaire à la paroi des ions en fonction de la coordonnée d'espace perpendiculaire au mur, il constate que pour que cette vitesse moyenne soit une fonction croissante, il faut qu'à l'entrée de la prégaine magnétique la vitesse moyenne perpendiculaire notée $u_{i,x}(0)$ satisfasse

$$u_{i,x}(0) \geqslant c_s \cos(\theta), \tag{1.4}$$

où $c_s = \sqrt{\frac{k_B(T_e+T_i)}{m_i}}$ désigne la vitesse acoustique ionique. Il constate également que la différence de potentiel entre l'entrée de la prégaine magnétique où le potentiel est supposé nul et l'entrée de la gaine noté Φ_s depend de l'angle θ et qu'elle satisfait

$$\left|\frac{q\Phi_s}{k_B T_e}\right| \propto |\ln(\cos(\theta))|. \tag{1.5}$$

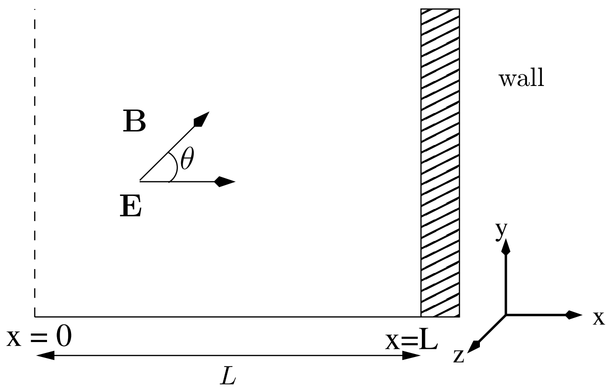


Figure 1.3 – Géométrie du modèle considéré par Chodura dans [19]. Le champ magnétique est constant et fait un angle θ avec la normale au mur.

Ces résultats ont été observés et validés numériquement dans d'autres travaux [57, 44, 17].

1.4.2 La problématique du champ magnétique tangent à la paroi

Une question naturelle que soulève l'analyse de Chodura concerne l'extension ou l'analyse de la physique lorsque le champ magnétique est parallèle à la paroi, c'est-à-dire faisant un angle $\theta = \frac{\pi}{2}$. Comme il le mentionne lui-même dans son article précurseur, son modèle dégénère dans ce cas, et l'estimation (1.5) n'est plus valide. Plusieurs tentatives d'explorations de la physique dans le cas $\theta = \frac{\pi}{2}$ ont été menées, notamment dans [62, 43, 32] où les auteurs considèrent le cas d'un champ magnétique qui n'est pas exactement parallèle à la paroi, mais rasant, c'est-à-dire où l'angle est $\theta = \frac{\pi}{2} - \eta$ où $\eta > 0$ est un angle petit. Dans ces travaux, les auteurs sont d'accord sur le fait que lorsque l'intensité du champ magnétique décroit la charge au mur décroit. Lorsque l'intensité du champ magnétique croît, la charge au mur croît mais le flux d'ions perpendiculaire au mur décroît. Dans le cas d'un champ magnétique parallèle au mur, Manfredi et Coulette dans [43] conjecturent le fait que la gaine pourrait ne pas exister. Une raison possible étant que pour un champ magnétique suffisamment fort, les particules seraient naturellement confinées à l'intérieur du plasma et la formation d'une gaine ne serait pas nécessaire pour confiner le plasma et préserver sa quasi neutralité. Selon Stangeby [63], il s'agit d'un problème ouvert que nous proposons d'étudier.

1.5 Objectif et démarche de la thèse: construction et simulation numérique de solutions stationnaires pour des systèmes de Vlasov-Poisson avec conditions de bords induisant l'ambipolarité

Dans cette thèse, on propose d'analyser mathématiquement le problème de la gaine en construisant des solutions stationnaires (c'est-à-dire indépendantes du temps) pour des systèmes de Vlasov-Poisson avec conditions de bords. La construction de ces solutions repose sur deux ingrédients importants: la construction d'invariants aux caractéristiques et une représentation explicites des fonctions de distributions, la minimisation de fonctionnelles d'énergies non linéaires associées à des équations de Poisson non linéaires. En particulier, ces fonctionnelles se décomposent sous la forme d'une énergie électrique, et d'un potentiel abstrait de type Sagdeev [58]. Le caractère bien posé et les propriétés qualitatives des solutions pour ce type d'équation ont été étudiés dans [3]. Les techniques de minimisation de fonctionnelles non linéaires pour des systèmes de Vlasov-Poisson ont déjà été utilisées dans [28, 37, 1, 15]. Les fonctionnelles que nous présentons sont des variantes de la fonctionnelle de Liouville [35]. L'existence de solutions stationnaires pour des systèmes de Vlasov-Poisson ou Vlasov-Maxwell avec conditions de bords a deja été établi auparavant dans le cas où une seule espèce est considérée [54, 53, 37]. Dans cette thèse, il s'agit aussi de déterminer les conditions limites pour que les solutions soient physiquement pertinentes, en un sens que nous précisons plus bas.

Pour le premier modèle que nous considérons, on cherche à determiner des conditions de bords pertinentes qui permettent de produire des solutions ayant toutes les propriétés physiques de la gaine de Debye, c'est-à-dire, un potentiel électrostatique décroissant, une densité de charge positive et l'équilibre des flux de charges à la paroi. Pour le deuxième modèle que nous présentons, on s'intéresse à l'étude de la stabilité linéaire d'un système de Vlasov-Ampère instationnaire autour de ses solutions stationnaires. Enfin, le dernier modèle que nous étudions s'articule autour de l'extension du premier modèle de Vlasov-Poisson au cas d'un champ magnétique constant. Des raisons techniques font que nous étudions le cas d'un champ magnétique tangent à la paroi qui est d'un point vue de la physique un problème ouvert. Pour le premier et le dernier modèle, les différents résultats obtenus seront illustrés par des simulations numériques avec des paramètres inspirés de la littérature.

1.5.1 Les systèmes de Vlasov-Poisson stationnaires

On adopte une approche cinétique de modélisation des plasmas. On notera $L > 0$ la taille caractéristique de la région de plasma modélisée, et on considère que le plasma est constitué d'une seule espèce d'ions. Pour chaque espèce de particules de symbole $s = i, e$ (i faisant référence aux ions et e aux électrons) nous décrivons le transport de cette espèce de particules par une fonction de densité notée F_s.

En l'absence de champ magnétique, cette fonction F_s dépend de la variable $X \in [0, L]$ qui désigne la position des particules et de la variable $V \in \mathbb{R}$ qui désigne leur vitesse. Cette fonction F_s positive peut-être interprétée comme la densité de probabilité de présence d'une particule de l'espèce s dans l'espace des phases $[0, L] \times \mathbb{R}$. La probabilité qu'une particule de l'espèce s se trouve dans un intervalle $(a, b) \subset [0, L]$ avec une vitesse en module inférieure à $v_{max} > 0$ est donnée par

$$\int_a^b \int_{\{V \in \mathbb{R}\ , |v| < v_{max}\}} F_s(X, V) dV dX.$$

Les positions et vitesses des particules de l'espèce s vérifieront alors les équations du mouvement

$$\begin{cases} (\mathcal{X}^s)'(t) = \mathcal{V}^s(t), \\ (\mathcal{V}^s)'(t) = \frac{q_s}{m_s} \partial_X \Phi(\mathcal{X}^s(t)). \end{cases}$$

où m_s désigne la masse de l'espèce s et $q_s = \begin{cases} q \text{ si } s = i, \\ -q \text{ si } s = e \end{cases}$ où q est la charge élémentaire. Le modèle sera qualifié de 1d-1v. Le potentiel électrostatique Φ est déterminé à partir de l'équation de Poisson

$$-\partial_{XX}\Phi = \frac{q}{\varepsilon_0}\left(\int_{\mathbb{R}} F_i(X,V)dV - \int_{\mathbb{R}} F_e(X,V)dV\right) \text{ dans } (0,L),$$

où ε_0 est la constante de permittivité du vide. La densité de particule de l'espèce s satisfait l'équation de Vlasov stationnaire

$$V\partial_X F_s - \frac{q_s}{m_s}\partial_X\Phi(X)\partial_V F_s = 0 \text{ dans } (0,L)\times\mathbb{R}.$$

En présence d'un champ magnétique externe que nous choisissons de la forme $\mathbf{B} := \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix}$ avec $B > 0$ donné, nous faisons l'hypothèse que la fonction de densité de l'espèce de particule s et le champ électrostatique sont invariants dans les plans parallèle au mur. La fonction F_s dépend donc de la variable $X \in [0,L]$ qui désigne la position des particules et de la variable $V \in \mathbb{R}^3$ qui désigne leur vitesse. Cette fonction F_s positive peut-être interprétée comme la densité de probabilité de présence d'une particule de l'espèce s dans l'espace des phases $[0,L]\times\mathbb{R}$. La probabilité qu'une particule de l'espèce s se trouve dans un intervalle $(a,b)\subset[0,L]$ avec une vitesse en module inférieure à $v_{max} > 0$ est donnée par

$$\int_a^b \int_{\{V\in\mathbb{R}^3\ ,|v|<v_{max}\}} F_s(X,V)dVdX.$$

Les positions et vitesses des particules de l'espèce s vérifieront alors les équations du mouvement

$$\begin{cases} (\mathcal{X}^s)'(t) = \mathcal{V}_x^s(t), \\ (\mathcal{V}_x^s)'(t) = \frac{q_s}{m_s}(-\partial_X\Phi(\mathcal{X}^s(t)) - B\mathcal{V}_z^s(t)), \\ (\mathcal{V}_y^s)'(t) = 0, \\ (\mathcal{V}_z^s)'(t) = \frac{q_s B}{m_s}\mathcal{V}_x^s(t). \end{cases}$$

Le modèle sera qualifié de 1d-3v. Le potentiel électrostatique Φ est déterminé à partir de l'équation de Poisson

$$-\partial_{XX}\Phi = \frac{q}{\varepsilon_0}\left(\int_{\mathbb{R}^3} F_i(X,V)dV - \int_{\mathbb{R}^3} F_e(X,V)dV\right) \text{ dans } (0,L).$$

La densité de particule de l'espèce s satisfait alors l'équation de Vlasov stationnaire

$$V_x\partial_X F_s + \frac{q_s}{m_s}\left(-\partial_X\Phi(X)\mathbf{e_x} + V\times\mathbf{B}\right)\cdot\nabla_V F_s = 0 \text{ dans } (0,L)\times\mathbb{R}^3.$$

où $\mathbf{e_x} = (1,0,0)^t$ et $V = (V_x,V_y,V_z)^t$.

1.5.2 Les conditions aux bords

Le bord gauche de l'intervalle $[0,L]$ $(X = 0)$ désigne le coeur du plasma et est une source de particules: on prescrit la densité de particules entrantes de l'espèce s par une fonction arbitraire notée F_s^{in},

$$F_s(0,V) = F_s^{in}(V) \text{ pour tout } V\in\mathbb{R}^d \text{ tel que } V_x > 0.$$

On impose également un potentiel électrostatique nul, c'est-à-dire $\Phi(0) = 0$. Le bord droit de l'intervalle ($X = L$) désigne le mur, on impose que les ions qui arrivent au mur sont complètement absorbés alors que les électrons sont partiellement ré-émis. Plus précisément, les conditions limites s'écrivent:

$$F_i(1, V) = 0 \text{ pour tout } V \in \mathbb{R}^d \text{ tel que } V_x < 0,$$

$$F_e(1, V) = \alpha F_e(1, -V_x, V_y, V_z) \text{ pour tout } V \in \mathbb{R}^d \text{ tel que } V_x < 0,$$

où $\alpha \in [0, 1]$ désigne la proportion d'électrons ré-émis. Quant au potentiel au mur, on écrit que $\Phi(1) = \Phi_w$, et on cherche à déterminer $\Phi_w \in \mathbb{R}$ de façon à ce que les courants d'ions et d'électrons au mur et dans la direction du mur s'égalisent, c'est-à-dire, tel que

$$\int_{\mathbb{R}^d} F_i(1, V) V_x dV = \int_{\mathbb{R}^d} F_e(1, V) V_x dV.$$

On remarque d'ailleurs qu'une intégration formelle en vitesse des équations de Vlasov montre que pour chaque espèce, la densité de courant dans la direction perpendiculaire au mur est constante en espace et donc pour tout $s = i, e$ et $X \in [0, L]$

$$\int_{\mathbb{R}^d} F_s(X, V) V_x dV = \int_{\mathbb{R}^d} F_s(1, V) V_x dV = \int_{\mathbb{R}^d} F_s(0, V) V_x dV.$$

On montre que ces densités de courants peuvent être exprimées en fonction des densités de particules entrantes et qu'il est possible sous certaines hypothèses d'en déduire le potentiel au mur Φ_w.

1.6 Résumé des travaux

Les chapitres de ce manuscrit sont composés des travaux suivants :

- Le chapitre 2 fait l'objet d'un article accepté [7].
- Le chapitre 3 fait l'objet d'un article soumis [6].
- Le chapitre 4 fait l'objet d'un article en préparation.

1.6.1 Chapitre 2. Etude d'un modèle de gaine bi-cinétique dans le cas purement électrostatique

Le chapitre 1 est consacré à l'étude d'un modèle de Vlasov-Poisson-Ampère unidimensionnel à deux espèces pour lequel nous recherchons des solutions correspondant aux gaines observées. En variables adimensionnées, les inconnues du modèles sont le potentiel électrostatique $\phi : [0, 1] \to \mathbb{R}$, la densité d'ions $f_i : [0, 1] \times \mathbb{R} \to \mathbb{R}^+$, la densité d'électrons $f_e : [0, 1] \times \mathbb{R} \to \mathbb{R}^+$, le potentiel au mur $\phi_w \in \mathbb{R}$ et la densité de référence au coeur du plasma $n_0 > 0$. Les inconnues vérifient le système

$$\begin{cases} v\partial_x f_e(x, v) + \dfrac{1}{\mu}\dfrac{d}{dx}\phi(x)\partial_v f_e(x, v) = 0 \quad \forall (x, v) \in (0, 1) \times \mathbb{R} \\ v\partial_x f_i(x, v) - \dfrac{d}{dx}\phi(x)\partial_v f_i(x, v) = 0 \quad \forall (x, v) \in (0, 1) \times \mathbb{R}, \\ -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = n_i(x) - n_e(x) \quad \forall x \in (0, 1), \\ \gamma_i(x) - \gamma_e(x) =: j(x) = 0 \quad \forall x \in [0, 1], \\ n_i(0) - n_e(0) = \rho_0, \end{cases}$$

où

$$n_i(x) := \int_{\mathbb{R}} f_i(x,v)dv, \quad n_e(x) := \int_{\mathbb{R}} f_e(x,v)dv,$$
$$\gamma_i(x) := \int_{\mathbb{R}} f_i(x,v)vdv, \quad \gamma_e(x) := \int_{\mathbb{R}} f_e(x,v)vdv,$$

avec les conditions limites

$$\begin{cases} f_e(0,v) = n_0 f_e^{in}(v) \quad \forall v > 0, \quad f_e(1,v) = \alpha f_e(1,-v) \quad \forall v < 0, \\ f_i(0,v) = f_i^{in}(v) \quad \forall v > 0, \quad f_i(1,v) = 0 \quad \forall v < 0, \\ \phi(0) = 0, \quad \phi(1) = \phi_w. \end{cases}$$

Ici, $\mu > 0$ désigne le rapport de masse entre les électrons et les ions, $\varepsilon > 0$ est la longueur de Debye normalisée et $\rho_0 \in \mathbb{R}$ est la charge en $x = 0$, f_i^{in} la densité d'ions entrants et f_e^{in} est une densité d'électrons entrants normalisé à un. La quatrième équation du modèle est l'équation de Maxwell-Ampère stationnaire en l'absence de champ magnétique. Nous commençons par fixer ϕ, ϕ_w et n_0 et étudions l'équation de Vlasov pour chaque espèce. A l'aide des caractéristiques et de l'invariant microscopique qu'est l'énergie d'une particule, on montre que si ϕ est décroissante alors les densités f_i et f_e sont déterminées en fonctions de ϕ par

$$f_i(x,v) = \begin{cases} f_i^{in}(\sqrt{v^2 + 2\phi(x)}) \text{ si } (x,v) \in \{(x,v) \in [0,1] \times \mathbb{R} \mid v > \sqrt{-2\phi(x)}\} \\ 0 \text{ si } (x,v) \in \{(x,v) \in [0,1] \times \mathbb{R} \mid v \leqslant \sqrt{-2\phi(x)}\}. \end{cases}$$

$$f_e(x,v) = \begin{cases} n_0 f_e^{in}\left(\sqrt{v^2 - \frac{2}{\mu}\phi(x)}\right) \text{ si } (x,v) \in \{(x,v) \in [0,1] \times \mathbb{R} \mid v \geqslant -\sqrt{\frac{2}{\mu}\left(\phi(x) - \phi_w\right)}\}, \\ \alpha n_0 f_e^{in}\left(\sqrt{v^2 - \frac{2}{\mu}\phi(x)}\right) \text{ si } (x,v) \in \{(x,v) \in [0,1] \times \mathbb{R} \mid v < -\sqrt{\frac{2}{\mu}\left(\phi(x) - \phi_w\right)}). \end{cases}$$

On détermine alors les densités macroscopiques n_i, n_e en fonction de ϕ ainsi que les densités de courants γ_i et γ_e qui sont respectivement données pour tout $x \in [0,1]$ par

$$\begin{aligned} n_i(x) &= \int_{\mathbb{R}^+} \frac{f_i^{in}(w)w}{\sqrt{w^2 - 2\phi(x)}}dv, \\ n_e(x) &= 2n_0 \int_{\sqrt{-\frac{2}{\mu}\phi(x)}}^{+\infty} \frac{f_e^{in}(v)v}{\sqrt{v^2 + \frac{2}{\mu}\phi(x)}}dv - n_0(1-\alpha)\int_{\sqrt{-\frac{2}{\mu}\phi_w}}^{+\infty} \frac{f_e^{in}(v)v}{\sqrt{v^2 + \frac{2}{\mu}\phi(x)}}dv, \\ \gamma_i(x) &= \int_{\mathbb{R}^+} f_i^{in}(v)vdv, \\ \gamma_e(x) &= (1-\alpha)n_0 \int_{\sqrt{-\frac{2}{\mu}\phi_w}}^{+\infty} f_e^{in}(v)vdv. \end{aligned}$$

Nous décidons de considérer dans la suite du travail, le cas où f_e^{in} est une Maxwellienne à l'entrée,

$$f_e^{in}(v) := \sqrt{\frac{2\mu}{\pi}} e^{-\dfrac{\mu v^2}{2}} \qquad \text{pour } v > 0.$$

On montre dans un premier temps que les paramètres α, μ et f_i^{in} étant choisis, le système

$$\begin{cases} n_i(0) - n_e(0) = \rho_0, \\ \gamma_i(x) = \gamma_e(x) \text{ pour tout } x \in [0,1], \end{cases} \tag{1.6}$$

ne dépend que de ϕ_w, n_0. On montre alors le résultat suivant.

Theorem 1.6.1. *Soit $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+;\mathbb{R}^+)$ tel que $\int_{\mathbb{R}^+} f_i^{in}(v)dv > \rho_0$. Il existe un unique couple $(n_0, \phi_w) \in (0,+\infty) \times \mathbb{R}^-$ dépendant uniquement de f_i^{in},α et μ vérifiant le système (1.6) si et seulement si*

$$0 < \frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0} \leqslant \frac{(1-\alpha)}{(1+\alpha)}\sqrt{\frac{2}{\mu\pi}} := s_1(\alpha).$$

De plus, ϕ_w est bornée inférieurement et on a

$$\ln\left(\frac{\sqrt{2\pi}}{(1-\alpha)\left(\sqrt{2} + \frac{1}{\sqrt{\mu}}\left(\frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv}\right)\right)}\right) \leqslant \phi_w \leqslant 0.$$

Par la suite, ce résultat permet d'éliminer les inconnues n_0 et ϕ_w et on se concentre alors sur l'équation de Poisson non linéaire avec densité Maxwellienne pour les électrons entrants (NLP-M)

$$\text{(NLP-M)} \ : \begin{cases} -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = (n_i - n_e)(x) \text{ for all } x \in (0,1) \\ \text{ avec les conditions limites de Dirichlet} \\ \phi(0) = 0 \text{ and } \phi(1) = \phi_w \\ \text{ où} \\ n_i(x) = \displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{\sqrt{v^2 - 2\phi(x)}}dv, \\ n_e(x) = \dfrac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi}e^{\phi(x)} - \dfrac{2n_0}{\sqrt{2\pi}}(1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}v}{\sqrt{v^2 + 2\phi(x)}}dv. \end{cases}$$

où f_i^{in} vérifie l'inégalité précédente de sorte que n_0 et ϕ_w sont entièrement déterminés à partir de l'équation de Maxwell-Ampère stationnaire et de l'équation de la charge à l'entrée du domaine. On propose un cadre fonctionnel permettant d'étudier (NLP-M) et on montre son caractère bien posé à travers l'étude d'une fonctionnelle d'énergie pour laquelle la solution de (NLP-M) est un minimiseur. Plus précisément, dans ce travail on considère des densités entrantes pour les ions intégrables, essentiellement bornées et d'énergies finies

$$\mathcal{I} := \left\{ h \in \left(L^1 \cap L^\infty\right)(\mathbb{R}^+;\mathbb{R}^+) \text{ tel que } \int_{\mathbb{R}^+} h(v)v^2dv < +\infty \right\}.$$

Puis pour $\rho_0 \in \mathbb{R}$ et $\alpha \in [0,1)$ donnés, on définit l'ensemble des densités d'ions entrants admissibles par

$$\mathcal{I}_{ad}(\rho_0,\alpha) := \left\{ h \in \mathcal{I} \text{ tel que } 0 < \frac{\displaystyle\int_{\mathbb{R}^+} h(v)v dv}{\displaystyle\int_{\mathbb{R}^+} h(v)dv - \rho_0} < s_1(\alpha) \right\},$$

et l'ensemble des potentiels admissibles par

$$V := V(\rho_0,\alpha) = \{\phi \in V_0 \mid \phi_w \leqslant \phi \leqslant 0 \text{ avec } \phi(1) = \phi_w\}, \tag{1.7}$$

où $V_0 := \{\phi \in H^1(0,1) \mid \phi(0) = 0\}$ est un espace de Hilbert equippé du produit scalaire $(\phi,\varphi)_{V_0} := \displaystyle\int_0^1 \phi'(x)\varphi'(x)dx$ pour tout $(\phi,\varphi) \in V_0 \times V_0$ et avec la norme induite définit par $\|\phi\|_{V_0} = \sqrt{(\phi,\phi)_{V_0}} = \|\phi\|_{H_0^1}$ pour tout $\phi \in V_0$. La fonctionnelle d'énergie que nous minimisons est définie pour tout $\phi \in V$ par

$$J_\varepsilon(\phi) := \int_0^1 \left(\frac{\varepsilon^2}{2}|\phi'(x)|^2 + \mathcal{U}(\phi(x)) \right) dx$$

où $\mathcal{U} : [\phi_w, 0] \to \mathbb{R}$ est donnée par

$$\mathcal{U}(\psi) := \int_{\mathbb{R}^+} f_i^{in}(v)v\sqrt{v^2 - 2\psi}dv + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi}e^{\psi} - (1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v\sqrt{v^2 + 2\psi}dv \right).$$

Enfin, on introduit $\alpha_c := \dfrac{1 - \sqrt{\frac{\pi\mu}{2}}}{1 + \sqrt{\frac{\pi\mu}{2}}}$ qui est un nombre compris entre 0 et 1 car pour tous les ions existants $\mu < \frac{2}{\pi}$. Le résultat principal de ce travail est le suivant.

Theorem 1.6.2. *Soit $\alpha \in [0,\alpha_c]$, $\rho_0 = 0$, $f_i^{in} \in \mathcal{I}_{ad}(0,\alpha)$ et $\varepsilon > 0$. Soit (n_0,ϕ_w) l'unique solution qui vérifie l'équation de Maxwell-Ampère stationnaire et l'équation de la charge à l'entrée (1.6). Supposons de plus le critère de Bohm cinétique*

$$\frac{\displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2}dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv} < \frac{\left(\sqrt{2\pi} + (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2}dv \right)}{\left(\sqrt{2\pi} - (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}}dv \right)}.$$

Alors le système de Vlasov-Poisson-Ampère est bien posé, avec une densité d'électron Maxwellienne à l'entrée. Plus précisément, il existe un unique $\phi_\varepsilon \in V$ solution de (NLP-M). De plus,

1. *Les densités f_e and f_i définies plus haut sont solutions faibles des équations de Vlasov.*
2. *Il existe $x^* \in [0,1)$ tel que sur $(x^*,1]$, ϕ est strictement décroissante et $n_i > n_e$.*
3. *Au mur, n_i, n_e, ϕ_w et les densités f_i, f_e ne dépendent pas de la longueur de Debye normalisée ε.*
4. *ϕ_ε est $C^2[0,1]$, concave, décroissante et nous avons les estimations quantitatives*

$$\|\phi_\varepsilon\|_{V_0} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) \quad \text{et} \quad \|n_i - n_e\|_{H^{-1}} = \mathcal{O}(\varepsilon). \tag{1.8}$$

L'introduction du nombre α_c vient de la difficulté à prouver l'existence de densités f_i^{in} admissibles qui vérifient le critère de Bohm cinétique. On sait prouver qu'il en existe pour $\alpha \leqslant \alpha_c$ mais pour $\alpha > \alpha_c$ la question reste en suspens. La neutralité à l'entrée $\rho_0 = 0$ et le critère de Bohm cinétique se traduisent sur la fonction $\mathcal{U}$ par le fait que $\mathcal{U}'(0) = 0$ et que $\mathcal{U}''(0) > 0$. Sous ces conditions, on sait montrer qu'alors la fonction $\mathcal{U}$ est décroissante sur $[\phi_w, 0]$ d'où l'on déduit la positivité de la charge $n_i - n_e$ dans $[0, 1]$. Les estimations quantitatives sur le champ électrique et sur la charge sont obtenues grâce au fait que la solution est un minimiseur de la fonctionnelle J_ε sur V. Nous menons par ailleurs une étude complémentaire sur les variations de la fonction $\mathcal{U}$ lorsque le critère de Bohm n'est pas vérifié. En outre, on montre que cette fonction $\mathcal{U}$ a au plus deux maxima locaux et deux minima locaux. On finit par illustrer les résultats obtenus à l'aide de simulations numériques en prenant un rapport de masse $\mu = \frac{1}{3672}$ qui correspond à un plasma de Deutérium. Nous considérons deux types de densités d'ions entrants: une vérifiant le critère de Bohm et l'autre non. Nous faisons varier les paramètres ε et α. L'observation commune est que pour ε petit, une couche limite chargée positivement de l'ordre de quelques ε en taille se forme proche du mur. Dans le cas où la densité d'ions à l'entrée ne satisfait pas le critère de Bohm, on observe la formation d'une deuxième couche limite à l'entrée mais qui est cette fois-ci chargée négativement.

1.6.2 Chapitre 3. Etude de la stabilité linéaire des électrons d'un modèle de gaine bi-cinétique

Le chapitre 3 est dédié à l'étude de la stabilité linéaire d'un équilibre solution du système étudié au chapitre 2. Les inconnues du modèle sont les densités $f_i : [0, +\infty) \times [0, 1] \times \mathbb{R} \to \mathbb{R}^+$, $f_e : [0, +\infty) \times [0, 1] \times \mathbb{R} \to \mathbb{R}^+$ et le champ électrique $E : [0, +\infty) \times [0, 1] \to \mathbb{R}$ qui satisfont le système de Vlasov-Ampère instationnaire suivant.

$$\begin{cases} \partial_t f_i + v\partial_x f_i + E\partial_v f_i = 0 & \text{in} \quad (0, +\infty) \times (0, 1) \times \mathbb{R}, \\ \partial_t f_e + v\partial_x f_e - \frac{1}{\mu} E\partial_v f_e = 0 & \text{in} \quad (0, +\infty) \times (0, 1) \times \mathbb{R}, \\ \varepsilon^2 \partial_t E = -j \text{ in } (0, +\infty) \times [0, 1], \quad j := \int_{\mathbb{R}} (f_i - f_e) v dv, \end{cases}$$

avec les conditions limites

$$\begin{cases} f_i(t, 0, v > 0) = f_i^{in}(v), \quad f_i(t, 1, v < 0) = 0, \\ f_e(t, 0, v > 0) = f_e^{in}(v), \quad f_e(t, 1, v < 0) = \alpha f_e(t, 1, -v), \end{cases}$$

où f_i^{in} et f_e^{in} sont des densités de particules entrantes. Comme dans le chapitre 2, on considère une densité d'électrons entrants Maxwellienne,

$$f_e^{in}(v) := n_0 \sqrt{\frac{2\mu}{\pi}} e^{-\dfrac{\mu v^2}{2}} \quad \text{pour } v > 0.$$

Les solutions de gaines que nous avons construites au chapitre 2, constituent pour ce système des solutions stationnaires. Pour des raisons pratiques dans la suite du travail les constantes μ et ε sont fixées à 1. On se pose la question de la stabilité d'un équilibre noté $(f_i^\infty, f_e^\infty, \phi^\infty)$ du système précédent, qui appartient à l'espace $(L^1 \cap L^\infty(\Omega))^2 \times C^2[0, 1]$ et vérifie les propriétés suivantes :

1. Pour tout $x \in [0, 1]$, $(\phi^\infty)''(x) \leqslant 0$, $E^\infty(x) := -(\phi^\infty)'(x) \geqslant 0$, $\phi^\infty(0) = 0$, $\phi^\infty(1) =: \phi_w$ and $E^\infty(1) =: E_w^\infty > 0$.

2. $f_i^\infty(x, v) = \begin{cases} f_i^{in}\left(\sqrt{v^2 + 2\phi^\infty(x)}\right) \text{ pour } (x, v) \text{ t.q } v > \sqrt{-2\phi^\infty(x)} \\ 0 \text{ ailleurs.} \end{cases}$

3. $f_e^\infty(x,v) = n_0\sqrt{\dfrac{2}{\pi}}\begin{cases} e^{-\frac{v^2}{2}}e^{\phi^\infty(x)} \text{ pour } (x,v) \text{ t.q } v \geqslant v_e(x) \\ \alpha e^{-\frac{v^2}{2}}e^{\phi^\infty(x)} \text{ pour } (x,v) \text{ t.q } v < v_e(x), \end{cases}$

4. $\int_{\mathbb{R}} f_i^\infty(x,v)vdv = \int_{\mathbb{R}} f_e^\infty(x,v)vdv$ pour tout $x \in [0,1]$.

Ici, la fonction v_e est définie pour tout $x \in [0,1]$ par $v_e(x) := -\sqrt{2\left(\phi^\infty(x) - \phi_w\right)}$. Comme la densité f_i^{in} ne peut pas être décroissante en raison du critère de Bohm cinétique établi au chapitre 2, la stabilité de l'équilibre ionique est plus délicat a étudié. On fait donc l'hypothèse qu'une fois à l'équilibre les ions sont fixes. Il s'agit également d'une simplification usuelle compte tenu des différences de mobilités entre les ions et les électrons. Par conséquent, on décide de perturber uniquement l'équilibre électronique et d'étudier la dynamique du système linéarisé autour d'une perturbation sur les électrons. Plus précisément, on écrit la solution du système de Vlasov-Ampère comme la somme de l'équilibre plus une perturbation intérieure sur les électrons et le champ électrique, c'est-à-dire $(f_i, f_e, \phi) = (f_i^\infty, f_e^\infty + \tilde{f}_e, \phi^\infty + \tilde{\phi})$. Le système de Vlasov-Ampère linéarisé (LVA) est alors donné par (en oubliant les ˜)

$$\text{(LVA):}\begin{cases} \partial_t f_e + Df_e = E\partial_v f_e^\infty, & \text{dans } (0,+\infty)\times\Omega \\ \partial_t E = \displaystyle\int_{\mathbb{R}} f_e v dv, & \text{dans } (0,+\infty)\times[0,1] \\ f_e(t,0,v>0) = 0, \quad f_e(t,1,v<0) = \alpha f_e(t,1,-v) & \text{dans } (0,+\infty) \end{cases}$$

où D désigne l'opérateur différentiel d'ordre un défini formellement par

$$D := v\partial_x - E^\infty\partial_v \text{ avec } E^\infty = -(\phi^\infty)' \text{ le champ électrique à l'équilibre.} \tag{1.9}$$

Comme f_e^∞ est discontinue le long de la courbe d'équation $v = v_e(x)$ sa dérivée en vitesse est donnée par

$$\partial_v f_e^\infty = [f_e^\infty]\delta^{v_e} - vf_e^\infty,$$

où δ^{v_e} est une distribution de Dirac supporté par la courbe d'équation $v = v_e(x)$, à savoir:

$$\langle\delta^{v_e}, \varphi\rangle = \int_0^1 \varphi(x, v_e(x))dx \quad \forall\varphi\in\mathcal{D}(\Omega),$$

et où

$$[f_e^\infty] := \lim_{v\to v_e(x)^+} f_e^\infty(x,v) - \lim_{v\to v_e(x)^-} f_e^\infty(x,v)$$

désigne le saut constant de f_e^∞ à travers la courbe d'équation $v = v_e(x)$ (la fonction f_e^∞ étant continue par morceaux, ces limites sont bien définies). Par conséquent, une idée naturelle est de rechercher les solutions de (LVA) se décomposant sous la forme d'une partie singulière et d'une partie régulière, c'est-à-dire,

$$f_e = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v)$$

où η_e et g_e sont deux fonctions. Le résultat principal de peut-être résumé de la façon suivante (en omettant de préciser les espaces fonctionnelles)

Theorem 1.6.3. *Pour toute donnée initiale (f_e^0, E^0) avec f_e^0 de la forme $f_e^0(x,v) = \eta^0(x)\delta^{v_e} + g_e^0(x,v)$ où η_0 et g_e^0 sont deux fonctions, il existe un couple de fonctions (η_e, g_e) et un champ électrique E tels que si on définit $f_e(t,x,v) = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v)$, alors le couple (f_e, E) est solution du système de Vlasov-Ampère linéarisé (LVA) avec la condition initiale $f_e(t=0,x,v) = f_e^0(x,v)$ et $E(t=0,x) = E^0(x)$.*

De plus, la fonctionnelle d'énergie positive définie pour tout $t \geqslant 0$ par

$$\mathcal{E}(t) = \frac{1}{2}\left(\int_0^1 \frac{\eta_e^2(t,x)|v_e(x)|dx}{[f_e^\infty]}dx + \int_\Omega \frac{g_e(t,x,v)^2}{f_e^\infty(x,v)}dxdv + \int_0^1 E(t,x)^2dx\right) \tag{1.10}$$

est décroissante.

Cette fonctionnelle d'énergie est une somme de carrés de normes L^2 à poids, il s'agit donc du carré d'une norme pour des triplets de fonctions (η_e, g_e, E) appartenant à un espace fonctionnelle adéquat. Le résultat que nous obtenons est un résultat de stabilité linéaire : partant d'une donnée initiale petite pour la norme de l'énergie, alors la norme de la solution du système linéarisé reste petite en tout temps. L'une des difficultés techniques vient du fait qu'à priori la fonction $w_e(t,x) := \eta_e(t,x)v_e(x)$ est solution d'une équation de transport dégénérée au bord (en $x = 1$) donnée par

$$\partial_t w_e(t,x) + v_e(x)\partial_x w_e(t,x) = [f_e^\infty]v_e(x)E(t,x) \text{ dans } (0,+\infty)\times(0,1).$$

Le champ de vitesse v_e est strictement négatif dans $(0,1)$ et nul en $x = 1$. Par ailleurs, sa dérivée n'est pas bornée sur $(0,1)$ puisque $\lim\limits_{x\to 1^-} v_e'(x) = +\infty$. La théorie de DiPerna-Lions [23] ne s'applique pas directement et nous proposons donc un cadre fonctionnel dans lequel il est possible de montrer l'existence d'une solution.

1.6.3 Chapitre 4. Etude d'un modèle de gaine bi-cinétique en présence d'un champ magnétique extérieur

Le chapitre 4 est consacré à l'étude d'un modèle de Vlasov-Poisson stationnaire unidimensionnel en espace et tridimensionnel en vitesse. Ce modèle est une extension du modèle du chapitre 2 au cas d'un champ magnétique constant et tangent à la paroi. Il s'inspire du modèle instationnaire utilisé par Manfredi et Coulette dans [43]. En variables adimensionnées, les inconnues du modèles sont le potentiel électrostatique $\phi : [0,1] \to \mathbb{R}$, la densité d'ions $f_i : [0,1]\times\mathbb{R}^3 \to \mathbb{R}^+$, la densité d'électrons $f_e : [0,1]\times\mathbb{R}^3 \to \mathbb{R}^+$, le potentiel au mur $\phi_w \in \mathbb{R}$ et la densité de référence au coeur du plasma $n_0 > 0$. Les inconnues vérifient le système

$$\begin{cases} v_x\partial_x f_i + (-\partial_x\phi\mathbf{e_x} + \omega_i\mathbf{v}\times\mathbf{b})\cdot\nabla_v f_i = 0 \quad \forall(x,\mathbf{v})\in(0,1)\times\mathbb{R}^3, \\ v_x\partial_x f_e - \dfrac{1}{\mu}(-\partial_x\phi\mathbf{e_x} + \omega_i\mathbf{v}\times\mathbf{b})\cdot\nabla_v f_e = 0 \quad \forall(x,\mathbf{v})\in(0,1)\times\mathbb{R}^3, \\ -\varepsilon^2\frac{d^2}{dx^2}\phi(x) = n_i(x) - n_e(x), \quad \forall x\in(0,1) \\ \gamma_i(x) - \gamma_e(x) := j(x) = 0, \quad \forall x\in[0,1], \\ n_i(0) - n_e(0) = 0. \end{cases}$$

où $\mathbf{e_x} := (1,0,0)^t$, $\mathbf{v} := (v_x, v_y, v_z)^t$, $\mathbf{b} := (0,1,0)^t$,

$$n_i(x) := \int_{\mathbb{R}^3} f_i(x,\mathbf{v})d\mathbf{v}, \quad n_e(x) := \int_{\mathbb{R}^3} f_e(x,\mathbf{v})d\mathbf{v},$$

$$\gamma_i(x) := \int_{\mathbb{R}^3} f_i(x,\mathbf{v})v_x d\mathbf{v}, \quad \gamma_e(x) := \int_{\mathbb{R}^3} f_e(x,\mathbf{v})v_x d\mathbf{v}$$

avec les conditions limites

$$\begin{cases} f_e(0,\mathbf{v}) = n_0 f_e^{in}(\mathbf{v})\,\forall\mathbf{v}\in(0,+\infty)\times\mathbb{R}^2, \ f_e(1,\mathbf{v}) = \alpha f_e(1,-v_x,v_y,v_z)\,\forall\mathbf{v}\in(-\infty,0)\times\mathbb{R}^2 \\ f_i(0,\mathbf{v}) = f_i^{in}(\mathbf{v}) \quad \forall\mathbf{v}\in(0,+\infty)\times\mathbb{R}^2, \quad f_i(1,\mathbf{v}) = 0 \quad \forall\mathbf{v}\in(-\infty,0)\times\mathbb{R}^2, \\ \phi(0) = 0, \quad \phi(1) = \phi_w. \end{cases}$$

Ici, $\mu > 0$ désigne le rapport de masse entre électrons et ions, $\omega_i > 0$ la fréquence de gyration des ions normalisée et ε la longueur de Debye normalisée. La quatrième équation du modèle traduit l'ambipolarité des flux dans la direction perpendiculaire au mur. On commence par fixer ϕ,ϕ_w et n_0 et on étudie l'équation de Vlasov linéaire pour chaque espèce. A l'aide des caractéristiques et de trois invariants microscopiques, on détermine les densités f_i,f_e, n_i,n_e en fonction de ϕ et les courants γ_i et γ_e (qui sont constants). Puis, on fait les hypothèses suivantes:

$$\phi \in W^{2,\infty}(0,1),\ \phi'' \leqslant \frac{\omega_i^2}{\mu} \text{ presque partout dans } (0,1) \text{ et pour tout } x \in [0,1]\ \phi(x) \leqslant 0,$$

$$\text{pour presque tout } (v_x, v_y, v_z) \in (0,+\infty) \times \mathbb{R} \times [-\frac{\omega_i}{2}, +\infty)\ f_i^{in}(v_x, v_y, v_z) = 0.$$

On se concentre ensuite sur le cas ou la densité d'électrons entrants est Maxwellienne,

$$f_e^{in}(\mathbf{v}) := 2\left(\frac{\mu}{2\pi}\right)^{\frac{3}{2}} e^{-\dfrac{\mu(v_x^2+v_y^2+v_z^2)}{2}} \quad \forall \mathbf{v} \in (0,+\infty) \times \mathbb{R}^2.$$

Ces hypothèses permettent de découpler du reste du système, l'équation d'ambipolarité et l'équation de la neutralité à l'entrée du reste du système. Plus précisément, on montre que les paramètres α, μ, ω_i et f_i^{in} étant choisis le système

$$\begin{cases} n_i(0) = n_e(0) \\ \gamma_i(x) = \gamma_e(x) \text{ pour tout } x \in [0,1] \end{cases}$$

ne dépend que de ϕ_w et n_0. On obtient alors le résultat suivant:

Theorem 1.6.4. *Soit $f_i^{in} \in L^1 \cap L^\infty\left(\mathbb{R}^+ \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2}); \mathbb{R}^+\right)$. Il existe un unique couple $(n_0, \phi_w) \in (0,+\infty) \times \mathbb{R}^-$ solution du système précédent si et seulement si*

$$0 < \frac{\int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) w_x dw_x dI_3 dI_2}{\int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) dw_x dI_3 dI_2} \leqslant s_1(\alpha, \omega_i, \mu), \tag{1.11}$$

où

$$s_1(\alpha, \omega_i, \mu) := \frac{2(1-\alpha)\,\mathrm{erfc}(\frac{\omega_i}{2\sqrt{2\mu}})}{\sqrt{2\pi\mu}\left((1+\alpha) + \frac{(1-\alpha)}{\sqrt{\pi}} \displaystyle\int_{-\infty}^{\frac{\omega_i}{2\sqrt{2\mu}}} e^{-\tilde{E_3}^2}\,\mathrm{erf}\left(\sqrt{\mu h\left(\sqrt{\frac{2}{\mu}}\tilde{E_3}, 0\right)}\right) d\tilde{E_3}\right)},$$

erf *désigne la fonction d'erreur définie pour tout x réel par*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

$\mathrm{erfc} = 1 - \mathrm{erf}$ *est la fonction d'erreur complémentaire et* $h(E_3, 0) = \frac{\omega_i^2}{2\mu^2} - \frac{E_3\omega_i}{\mu}$.

En particulier, on remarque que dans le cas limite $\omega_i = 0$ on retrouve le même résultat qu'au chapitre 2. Par ailleurs, on observe que les hypothèses précédentes ont une conséquence directe sur le type de scénarios physiques que l'on considère. Plus particulièrement, les ions qui entrent (en $x = 0$) avec des vitesses $v_x > 0$ et $v_z < -\frac{\omega_i}{2}$ sont nécessairement destinés à toucher le mur (en $x = 1$). Pour les électrons, on introduit la borne positive $E(\phi_w) := \frac{1}{\omega_i}\left(\frac{\omega_i^2}{2\mu} - \phi_w\right)$ et on décrit deux types scénarios possibles: soit ils entrent (en $x = 0$) avec des vitesses $v_x > 0$ et $v_z \geqslant E(\phi_w)$ et ils sont destinés à atteindre le mur (en

$x = 1$), soit ils entrent avec des vitesses $v_x > 0$ et $v_z < E(\phi_w)$ et dans ce cas, soit ils atteignent le mur car v_x est suffisamment grand, soit ils ne l'atteignent pas et reviennent (en $x = 0$) avec une vitesse $v_x < 0$. Cela empêche que les particules soient naturellement confinées par le champ magnétique même avec un champ électrique nul. On se concentre par la suite sur le problème de Poisson non linéaire (NLP-MMAG)

$$\text{(NLP-MMAG):} \begin{cases} \text{Soit } \alpha \in [0,1), \omega_i > 0, \mu > 0, f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha, \omega_i, \mu) \text{ et } \varepsilon > 0, \\ \text{trouver } \phi \in V_{ad}(\alpha, \omega_i, \mu) \\ -\varepsilon^2 \phi''(x) = \rho_i(x, \phi(x)) - \rho_e(x, \phi(x)) \text{ pour tout } x \in (0,1). \end{cases}$$

où $\mathcal{I}_{ad}^{in}(\alpha, \omega_i, \mu)$ désigne l'ensemble des fonctions intégrables, essentiellement bornés, d'énergies finies sur $\mathbb{R}^+ \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2})$ et vérifiant l'inégalité (1.11). Ces fonctions sont admissibles car pour chacune d'elle il existe un unique $(n_0, \phi_w) \in (0, +\infty) \times (-\infty, 0]$ qui vérifie l'équation d'ambipolarité et la neutralité de la charge à l'entrée. L'espace des potentiels admissibles est donné par l'ensemble convexe et fermé (dans V_0 avec la norme $H_0^1(0,1)$)

$$V_{ad}(\alpha, \omega_i, \mu) := \left\{ \phi \in V_0 \text{ tel que } \phi(1) = \phi_w \text{ et } \underline{q}(x) \leqslant \phi(x) \leqslant 0 \, \forall x \in [0,1] \right\},$$

où

$$\underline{q}(x) := \frac{-\omega_i^2 x(1-x)}{2\mu} + x\phi_w \quad \forall x \in [0,1],$$

et

$$V_0 := \{\phi \in H^1(0,1) \text{ tel que } \phi(0) = 0\}$$

est un espace de Hilbert équipé du produit scalaire défini pour tout $(\varphi, \psi) \in V_0^2$ par $(\varphi, \psi)_{V_0} := \int_0^1 \varphi'(x)\psi'(x)dx$. Comme au chapitre 2, les fonctions ρ_i et ρ_e qui représentent la charge respective des ions et des électrons dépendent du potentiel ϕ mais aussi de la variable x à cause de la présence du champ magnétique. On décide de rechercher les solutions de (NLP-MMAG) comme des minimiseurs sur $V_{ad}(\alpha, \omega_i, \mu)$ d'une fonctionnelle d'énergie qui est définie par

$$J_\varepsilon(\phi) := \int_0^1 \frac{\varepsilon^2}{2} |\phi'(x)|^2 - \mathcal{U}(x, \phi(x)) dx \quad \forall \phi \in V_{ad}(\alpha, \omega_i, \mu).$$

où $\mathcal{U}(x, .)$ est une primitive de $\rho_i(x, .) - \rho_e(x, .)$. Nous obtenons alors le résultat d'existence de minimiseurs pour J_ε.

Theorem 1.6.5. *Soit $\alpha \in [0,1), \omega_i > 0, \mu > 0, f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha, \omega_i, \mu)$ et $\varepsilon > 0$. Il existe $\phi_\varepsilon \in V_{ad}(\alpha, \omega_i, \mu)$ tel que*

$$J_\varepsilon(\phi_\varepsilon) \leqslant J_\varepsilon(\psi) \quad \forall \psi \in V_{ad}(\alpha, \omega_i, \mu).$$

De plus, on a

$$\|\phi_\varepsilon\|_{V_0} = \mathcal{O}(\frac{1}{\varepsilon}).$$

L'existence de solution pour (NLP-MMAG) reste un problème ouvert. Nous montrons toutefois que même en l'absence de résultats mathématiques définitifs, ce modèle peut se discrétiser ce qui permet d'obtenir des solutions numériques pertinentes (il y aura cependant des restrictions qui seront discutées au chapitre 4). Nous donnons quelques résultats de simulations numériques pour une certaine plage de paramètres. Plus particulièrement, nous considérons $\mu = \frac{1}{3672}$ qui correspond à un plasma de Deuterium et fixons $\alpha = 0$. Nous effectuons des simulations pour deux densités f_i^{in} différentes. Dans les deux cas, on fait varier l'intensité du champ magnétique ω_i de 0 à une valeur critique $\omega_i^c > 0$ qui donne l'égalité dans l'inégalité (1.11). On remarque dans les deux cas, que lorsque l'intensité du champ magnétique augmentent la valeur absolue du potentiel au mur $|\phi_w|$ diminue. Le charge relative au mur

$(\rho_i - \rho_e)(1, \phi_w)$ est toujours positive et est une fonction croissante de ω_i. De plus, quand ω_i augmente le flux d'ions diminue. On représente également le potentiel électrostatique ϕ_ε et la densité de charge $x \mapsto \rho_i(x, \phi_\varepsilon(x)) - \rho_e(x, \phi_\varepsilon(x))$ pour une valeur donnée ε et pour plusieurs valeurs de ω_i. Enfin, on représente les trajectoires de trois ions et trois électrons dans l'espace tridimensionnel pour la plus grande valeur du champ magnétique. On illustre en particulier les différents scénarios physiques attendus pour des f_i^{in} admissibles. Les ions qui entrent en $x = 0$ avec des vitesses $v_x > 0$ et $v_z < -\frac{\omega_i}{2}$ atteignent toujours $x = 1$. Pour les électrons, deux d'entre eux atteignent la paroi alors que l'autre revient en $x = 0$. Enfin, on fait le lien avec les résultats déjà existants.

1.7 Le plan

Cette thèse est composé de trois chapitres. Chaque chapitre contient une introduction et une conclusion.

Dans le premier chapitre, on analyse le modèle de gaine basé sur un système de Vlasov-Poisson à deux espèces, unidimensionnelle (1d-1v), en présence de conditions limites. Dans la section 2.2 on détaille la structure mathématique du modèle. En section 2.3, on donne des hypothèses sous lesquels notre problème est équivalent un problème de Poisson non linéaire. Puis, en section 2.4, on détermine les conditions limites de façon à ce que les solutions stationnaires du système vérifient l'equation d'Ampère et l'équation de la charge à l'entrée. On montre le caractère bien posé du modèle dans la section 2.6. Dans la section 2.7 on présente brièvement les méthodes numériques implémentées et on donne des résultats de simulations numériques.

Le deuxième chapitre concerne la stabilité linéaire des équilibres de gaines. Dans la section 3.1, on introduit un modèle instationnaire de type Vlasov-Ampère pour lequel les solutions de gaines du premier chapitre sont des équilibres. En section 3.2, on linéarise le système autour d'un équilibre en supposant les ions fixes et on donne l'énoncé précis du résultat de stabilité. Dans la section 3.3, on démontre une inégalité de type Hardy-Poincaré et on on démontre le résultat de stabilité. On discute brièvement la régularité de la solution en section 3.4.

Le troisième chapitre est dédié à l'étude d'un système de Vlasov-Poisson à deux espèces, unidimensionnelle en espace et tridimensionnel en vitesse (1d-3v), en présence de conditions limites et avec un champ magnétique constant et tangent à la paroi. Dans la section 4.2 on présente et détaille la structure mathématique du modèle.f the model and write down its mathematical structure. En section 4.3.1, on montre sous certaines hypothèses l'équivalence entre le modèle et une équation de Poisson non linéaire. Puis dans la section 4.4, on détermine des conditions limites pour lesquelles les solutions stationnaires vérifient l'ambipolarité des flux dans la direction perpendiculaire au mur et l'équation de neutralité à l'entrée. On donne un cadre fonctionnel et on étudie le problème de minimisation associé à l'équation de Poisson non linéaire en section 4.5. On décrit les méthodes numériques implémentées, on donne des résultats de simulations numériques et on fait le lien avec les résultats existants en section 4.6.

Enfin, cette thèse s'achève par une conclusion et la présentation de quelques perspectives.

Chapter 2

A bi-kinetic model of plasma sheath: the electrostatic case

2.1 Introduction

The description of the plasma-wall interaction is a challenging issue with many practical applications, be it in the modeling of Tokamak walls or ionic engines for satellites. Thus, the mathematical study and the numerical simulation of physically consistent models is of interest. When a plasma is in contact with an isolated and partially absorbing wall, a thin net-charge layer develops spontaneously between the wall and the plasma. This layer of several Debye lengths is called a sheath [12, 18, 63] and it is usually understood as the way by which the plasma preserves its global neutrality. Indeed, because the electrons are a lighter species they are prone to exit the plasma at a higher rate than the heavier ions. As this phenomenon alone would result in an unstable situation, namely a positive charge built up in the core plasma, the negative charge accumulated at the isolated wall causes the electric potential to drop and repel a significant fraction of the electrons. The magnitude of the drop is then such that the flow is ambipolar, in the sense that positive and negative charges exit the core plasma at the same rate [18, 63].

Plasma-sheaths have been extensively studied in the last decades [12, 40, 18, 64, 59], however several important questions do not have fully satisfactory answers on the mathematical level. For instance, we are not aware of a simple model that describes in a unified way the physical processes at play between the sheath and the core of the plasma. Nevertheless, a common observation that is supported by both theoretical and empirical evidence is that at the sheath entrance the average ion velocity must exceed its sound speed c_s,

$$u_i > c_s := \sqrt{\frac{kT_e}{m_i}} \tag{2.1}$$

where k is the Boltzmann constant, T_e is the electronic temperature and m_i is the ion mass [18]. This definition of the ion sound velocity corresponds to the case where the ion temperature T_i is much smaller than the electron one. Another possible definition for the ion sound velocity is $c_s' := \sqrt{\dfrac{k(T_e + T_i)}{m_i}}$. This inequality is often referred to as the original *Bohm criterion* and several variants have been developed in the scope of more general models [57, 5]. For instance, in the case of a plasma consisting of a Poisson equation to define the electrostatic potential ϕ coupled to differential equations to define the ion and electron density n_i and n_e, it has been shown that these densities can both be expressed as functions of ϕ, and that at the sheath entrance (which is commonly defined as the limit between the non neutral region and the neutral region), the value ϕ_{se} of the potential must be such that

$$\frac{d}{d\phi}\left(n_i - n_e\right)\left(\phi_{se}\right) \leqslant 0. \tag{2.2}$$

The sheath-edge x_{se}, namely the entrance of the sheath, is then often defined as the position where $\phi(x_{se}) = \phi_{se}$, even though it is commonly admitted that the sheath-edge is a difficult place to define [57]. Overall, the inequality expresses the idea that at the entrance of the sheath the electron density decreases more rapidly than the ion density as the electric potential drops.

As for the boundary condition on the wall, most models describe the potential as having a "floating" value that adjusts itself according to the dynamics of the system. However no clear definition of a self consistent wall potential seems available. On the mathematical side some models have been proposed but they do not fully answer the above questions, see e.g. [42, 25, 26, 21].

In the present work we address this problem by considering a simple plasma-wall interaction model with a self-consistent potential and we show that it is well posed under the assumption that the incoming ion distribution satisfies a moment condition which generalizes the usual kinetic Bohm criterion. Moreover, our solutions share most of the properties of plasma sheaths, such as a decreasing potential and a positive charge density. In our model the ion and electron densities are solutions to one dimensional stationary Vlasov equations coupled with a self consistent Poisson equation. Boundary conditions are determined to reflect the physical properties of this simplified model: in particular, the wall potential is determined so that the Ampère equation holds for the stationary solutions. A surprising result is that the resulting potential is only well-defined for incoming ions satisfying an *upper bound* on their average velocity. This constraint is shown to be compatible with the Bohm criterion thanks to the large mass ratio between ions and electrons.

To allow some generality, we consider that electrons are re-emitted with probability $\alpha \leq 1$ while ions are totally absorbed. Ions and electrons are assumed to enter the plasma with given velocity distributions. Since the core of the plasma is well described by a full Maxwellian, we have chosen to consider (semi-) Maxwellian distributions for the incoming electrons. At the numerical level we then observe that the resulting velocity distribution is very close to a full Maxwellian when far from the wall, in good qualitative agreement with the results from [63, p. 75].

The plan of this chapter is organized as follows. In section 2.2 we begin with a presentation of the model and write down its mathematical structure. Then in section 2.3, we show under a decreasing assumption on the electrostatic potential, the equivalence between the Vlasov-Poisson system and a non linear Poisson equation. In section 2.4, we prove that we can determine a priori the floating potential by solving a non linear equation. As a result, a necessary and sufficient condition for the wall potential to be uniquely determined is established. We also give an a priori lower bound for the wall potential. In section 2.6 we set up the mathematical framework that is used in the rest of the paper and state the main result (well posedness under a condition that generalizes the usual kinetic Bohm criterion, and quasi-neutrality estimates). The proof relies on reformulating the non linear Poisson equation as a minimization problem, and our generalized Bohm inequality appears naturally as a local convexity condition for the energy functional. A stronger variant of the kinetic Bohm criterion is also provided in the end of the section. Especillay, it is a sufficient condition for our energy functional to be strictly convex. Finally, in section 2.7, we describe the numerical methods employed to solve the problem. Then we illustrate the main result with a physically based sheath problem and present additional results when the kinetic Bohm criterion is not fulfilled. Final comments about the range of applicability of this work are provided as a conclusion in section 2.8.

2.2 Description of the bi-kinetic model

2.2.1 Physical setting

We consider a plasma at equilibrium made of one species of ions and electrons. This plasma is assumed to be contained in a one dimensional chamber. This model only describes a portion of the chamber of length $L > 0$. Physical quantities will often be denoted with upper case while normalized ones will be denoted with lower case. Our system is subject to the following physical considerations:

1. The plasma is assumed to be non-collisional.

2. The effect of the self-consistent magnetic field is neglected.

3. The physical quantities that describe the plasma state such as, the ionic distribution, the electronic distribution and the electric potential (that we will denote F_i,F_e and Φ) depend (in space) exclusively on the longitudinal variable denoted X.

4. At $X = 0$ we consider:

   (a) that the potential Φ is arbritrarily set to zero;

   (b) ions and electrons entering the domain with positive velocities which are described through non negative velocity distributions denoted respectively F_i^{in} and F_e^{in};

   (c) an arbitrary charge imbalance denoted P_0 (normalized value ρ_0);

5. At the wall, that is at $X = L$ (or $x = 1$ in normalized variables), we consider:

   (a) purely absorbing conditions for ions, i.e, ions are not re-emitted from the wall;

   (b) electrons re-emitted with probability $\alpha \in [0, 1]$.

   (c) no net current.

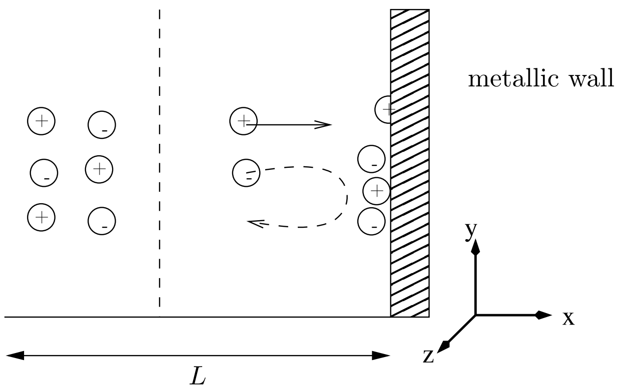


Figure 2.1 – Schematic illustration. Ions and electrons are going toward the wall, some electrons reaching the wall are re-emitted with a probability $\alpha \in [0, 1]$ while ions are totally absorbed.

2.2.2 Kinetic modeling of the stationary plasma wall interaction

We first write the equations in physical variables and then derive a dimensionless model. In this model, the unknowns functions are respectively the electric potential $\Phi : [0, L] \to \mathbb{R}$, the ion distribution function $F_i : [0, L] \times \mathbb{R} \to \mathbb{R}^+$, the electron distribution function $F_e : [0, L] \times \mathbb{R} \to \mathbb{R}^+$. We shall also consider two degrees of freedom: the reference electron density $N_0 > 0$ that is a parameter of the incoming electron boundary condition and the wall potential $\Phi_W \in \mathbb{R}$ that is a boundary condition for the electric potential Φ. We denote by $N_i(X) := \int_{\mathbb{R}} F_i(X, V)dV$ (respectively $N_e(X) := \int_{\mathbb{R}} F_e(X, V)dV$) the ionic (respectively the electronic) density at $X \in [0, L]$ and $\Gamma_i(X) := \int_{\mathbb{R}} F_i(X, V)V dV$ (respectively $\Gamma_e(X) := \int_{\mathbb{R}} F_e(X, V)V dV$) the ionic (respectively the electronic) flux at $X \in [0, L]$. The equations governing the ion and electron transport in the plasma, with an electric field $E = -\frac{d}{dX}\Phi$ are assumed to be stationary Vlasov equations and write

$$V\partial_X F_e(X, V) + \frac{q}{m_e}\frac{d}{dX}\Phi(X)\partial_V F_e(X, V) = 0 \quad \forall(X, V) \in (0, L) \times \mathbb{R}, \tag{2.3}$$

$$V\partial_X F_i(X, V) - \frac{q}{m_i}\frac{d}{dX}\Phi(X)\partial_V F_i(X, V) = 0 \quad \forall(X, V) \in (0, L) \times \mathbb{R}, \tag{2.4}$$

with the boundary conditions

$$F_e(0, V) = N_0 F_e^{in}(V) \qquad \text{for } V > 0, \tag{2.5}$$

$$F_e(L, V) = \alpha F_e(L, -V) \qquad \text{for } V < 0, \tag{2.6}$$

$$F_i(0, V) = F_i^{in}(V) \qquad \text{for } V > 0, \tag{2.7}$$

$$F_i(L, V) = 0 \qquad \text{for } V < 0, \tag{2.8}$$

here F_e^{in},F_i^{in} are velocity distributions that represent the way particles are injected, $q > 0$ is the electric charge and $m_i > 0$ (respectively $m_e > 0$) denotes the ionic (respectively the electronic) mass. A formal integration of equations (2.3)-(2.4) with respect to the velocity variable shows that the current density defined for all $X \in [0, L]$ by $J(X) := q\,(\Gamma_i(X) - \Gamma_e(X))$ has to be constant in space, and so $J(X) = J(L) = J(0)$ for all $X \in [0, L]$. We consequently require that there is no net current density

$$J(X) = 0 \quad \forall X \in [0, L], \tag{2.9}$$

since the ambipolarity in the sheath require that the flux of charges at the wall is zero. The electric potential is determined from the densities through the Gauss law

$$-\frac{d^2}{dX^2}\Phi(X) = \frac{q}{\varepsilon_0}\,(N_i(X) - N_e(X)) \quad \forall X \in (0, L) \tag{2.10}$$

with boundary conditions

$$\Phi(0) = 0, \qquad \Phi(L) = \Phi_W, \tag{2.11}$$

where ε_0 is the vacuum permittivity. Lastly, the equation on the charge imbalence at the entrance of the domain writes

$$N_i(0) - N_e(0) = P_0. \tag{2.12}$$

For the mathematical analysis of this model, it is convenient to rescale the equations and to this end we introduce the dimensionless variables x, v and the dimensionless functions ϕ, f_i and f_e defined as:

$$x := \frac{X}{L}, \quad v := \frac{V}{c_s},$$

$$f_i(x,v) := Lc_sF_i(Lx, c_sv), \quad f_e(x,v) := Lc_sF_e(Lx, c_sv), \quad \phi(x) := \frac{q}{kT_e}\Phi(Lx),$$

where k_B is the Boltzmann constant, T_e is a reference electron temperature, $c_s := \sqrt{\frac{k_BT_e}{m_i}}$ the ion sound speed and $\mu := \frac{m_e}{m_i}$ the mass ratio. We also define the dimensionless quantities

$$n_i(x) := \int_{\mathbb{R}} f_i(x,v)dv, \quad n_e(x) := \int_{\mathbb{R}} f_e(x,v)dv,$$

$$\gamma_i(x) := \int_{\mathbb{R}} f_i(x,v)vdv, \quad \gamma_e(x) := \int_{\mathbb{R}} f_e(x,v)vdv,$$

$$n_0 := LN_0, \quad \rho_0 := LP_0, \qquad \phi_w := \frac{q}{kT_e}\Phi_W,$$

$$f_e^{in}(v) := c_sF_e^{in}(c_sv), \quad f_i^{in}(v) := Lc_sF_i^{in}(c_sv).$$

The coupled boundary value problem (2.3)-(2.12) is then equivalent to the following boundary value problem:

$$\begin{cases} v\partial_x f_e(x,v) + \dfrac{1}{\mu}\dfrac{d}{dx}\phi(x)\partial_v f_e(x,v) = 0 \quad \forall (x,v) \in (0,1)\times\mathbb{R}, & (2.13)\\ v\partial_x f_i(x,v) - \dfrac{d}{dx}\phi(x)\partial_v f_i(x,v) = 0 \quad \forall (x,v) \in (0,1)\times\mathbb{R}, & (2.14)\\ -\varepsilon^2\dfrac{d^2}{dx^2}\phi(x) = n_i(x) - n_e(x) \quad \forall x \in (0,1), & (2.15)\\ \gamma_i(x) - \gamma_e(x) =: j(x) = 0 \quad \forall x \in [0,1], & (2.16)\\ n_i(0) - n_e(0) = \rho_0. & (2.17) \end{cases}$$

complemented with the boundary conditions

$$\begin{cases} f_e(0,v) = n_0 f_e^{in}(v) \quad \forall v > 0, \quad f_e(1,v) = \alpha f_e(1,-v) \quad \forall v < 0, & (2.18)\\ f_i(0,v) = f_i^{in}(v) \quad \forall v > 0, \quad f_i(1,v) = 0 \quad \forall v < 0, & (2.19)\\ \phi(0) = 0, \quad \phi(1) = \phi_w & (2.20) \end{cases}$$

Here, we have set $\varepsilon := \lambda_D\sqrt{\frac{N_0}{L}}$ where $\lambda_D := \sqrt{\frac{\varepsilon_0 kT_e}{q^2N_0}}$ is the Debye length. The parameter ε is a normalized Debye length that does not depend on N_0. This model contains five unknowns f_i, f_e, ϕ, ϕ_w and n_0 and five sets of equations: the two Vlasov problem, the Poisson equation, the Ampère equation and the charge imbalance relation. The unknowns ϕ, f_i, f_e and ϕ_w can be seen as state-variables of our physical system, in the sense that they are determined by fundamental physical laws. The unknown n_0

is more like a parameter to be adjusted to ensure the prescribed charge at $x = 0$. This not properly said, fundamental. The introduction of the unknown n_0 enables us to get rid of a possible constrain on the incoming boundary conditions. The set of equations (2.13)-(2.17) is the model problem and we will refer to it as the Vlasov-Poisson-Ampère problem. It contains the main physical parameters ε, α, ρ_0, μ, f_e^{in} and f_i^{in}. The Vlasov-Poisson problem is made of equations (2.13)-(2.15), (2.18)-(2.20) which can be considered as the main equations while the Ampère equation (2.16) and the charge imbalance relation (2.17) can be considered as additional physical constraints. To our knowledge, this stationary and bi-kinetic boundary-value problem has never been studied in full details. For example in [54], a model of plane diode is studied. It is consists of a one single stationary Vlasov equation for electrons coupled with the Poisson equation for the electrostatic potential, the well-posedness is studied for a large class of electron boundary conditions. In [31], the non-stationary version of the plane diode is studied. In the two dimensional case, stationary solutions to a Vlasov-Poisson system in a polygonal domain have already been constructed in the work of [37]. In particular, the approach followed consists in constructing stationary solutions as minimizers of an energy functional. Our work clearly follows the same idea.

2.3 Reformulation as a non linear Poisson problem

Thanks to the one-dimensional structure of the Vlasov-Poisson problem (2.13)-(2.20), it is possible to reformulate it as a non linear Poisson equation. When the potential ϕ is given both Vlasov equations for ions and electrons are linear transport equations, and their solutions are determined by transport along the characteristics of the (incoming) boundary conditions. The main result of this section is the following.

Proposition 2.3.1 (Equivalence)**.** *Let $n_0 > 0$, $\phi_w < 0$ and $\phi \in W^{2,\infty}(0,1)$ be such that $\phi' < 0$ with $\phi(0) = 0$ and $\phi(1) = \phi_w$. Assume moreover that $(f_i^{in}, f_e^{in}) \in L^1 \cap L^\infty(\mathbb{R}^+)^2$. Then the following are equivalent: a) (f_i, f_e, ϕ) is solution to the Vlasov-Poisson system with*

$$f_i(x,v) = \begin{cases} f_i^{in}(\sqrt{v^2 + 2\phi(x)}) \ \textit{if} \ (x,v) \in \{(x,v) \in [0,1]\times\mathbb{R} \mid v > \sqrt{-2\phi(x)}\} \\ 0 \ \textit{if} \ (x,v) \in \{(x,v) \in [0,1]\times\mathbb{R} \mid v \leqslant \sqrt{-2\phi(x)}\}. \end{cases}$$

$$f_e(x,v) = \begin{cases} n_0 f_e^{in}\left(\sqrt{v^2 - \frac{2}{\mu}\phi(x)}\right) \ \textit{if} \ (x,v) \in \{(x,v) \in [0,1]\times\mathbb{R} \ \mid \ v \geqslant -\sqrt{\frac{2}{\mu}\left(\phi(x) - \phi_w\right)}\}, \\ \alpha n_0 f_e^{in}\left(\sqrt{v^2 - \frac{2}{\mu}\phi(x)}\right) \ \textit{if} \ (x,v) \in \{(x,v) \in [0,1]\times\mathbb{R} \mid v < -\sqrt{\frac{2}{\mu}\left(\phi(x) - \phi_w\right)}). \end{cases}$$

b) ϕ is a solution to

$$(NLP)\ :\ \begin{cases} -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = (n_i - n_e)(x) \quad \textit{for a.e } x \in (0,1) \\ \phi(0) = 0, \quad \phi(1) = \phi_w \\ \ \textit{with} \\ n_i(x) = \displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{\sqrt{v^2 - 2\phi(x)}} dv \\ n_e(x) = 2n_0 \displaystyle\int_{\sqrt{-\frac{2}{\mu}\phi(x)}}^{+\infty} \frac{f_e^{in}(v)v}{\sqrt{v^2 + \frac{2}{\mu}\phi(x)}} dv \\ \qquad -n_0(1-\alpha) \displaystyle\int_{\sqrt{-\frac{2}{\mu}\phi_w}}^{+\infty} \frac{f_e^{in}(v)v}{\sqrt{v^2 + \frac{2}{\mu}\phi(x)}} dv. \end{cases} \tag{2.21}$$

The proof of the above proposition relies on the two main ingredients that are:

a) The conservation of the particle energy along the characteristic curves.

b) The explicit representation of the distribution functions.

2.3.1 Study of the linear Vlasov system

As a preliminary step, it is natural to consider the electrostatic potential ϕ as given. Consequently, in this section we assume to be given $n_0 > 0$, $\phi_w < 0$ and $\phi \in W^{2,\infty}(0,1)$. We also consider, as in a Debye sheath, that the electrostatic potential is decreasing, that is $\phi' < 0$ in $[0,1]$ with $\phi(0) = 0$ and $\phi(1) = \phi_w$. The regularity of ϕ is sufficient to ensure that the characteristic curves are well-defined, see [23].

Study of the electron characteristics

Definition 2.3.2. *The characteristics trajectories of electrons* (2.13) *are the curves which satisfy the ordinary differential system of equations*

$$(C_e): \begin{cases} \dot{\mathcal{X}}(t) = \mathcal{V}(t) \\ \dot{\mathcal{V}}(t) = \frac{1}{\mu}\frac{d}{dx}\phi(\mathcal{X}(t)) \\ \mathcal{X}(0) = x \\ \mathcal{V}(0) = v \end{cases}$$

for an arbitrary initial data $(x,v) \in [0,1] \times \mathbb{R}$. *For any arbitrary initial data* $(x,v) \in (0,1) \times \mathbb{R} \cup (\{0\} \times [0,+\infty)) \cup (\{1\} \times (-\infty, 0])$ *there is a unique solution denoted* $(\mathcal{X}(t;x,v), \mathcal{V}(t;x,v))$ *for all* $t \in [t_{in}(x,v), t_{out}(x,v)]$ *where*

$$t_{in}(x,v) := \inf\{\tau \leqslant 0 : \mathcal{X}(s,x,v) \in (0,1), \forall s \in (\tau,0)\},$$

$$t_{out}(x,v) := \sup\{\tau \geqslant 0, \mathcal{X}(s,x,v) \in (0,1), \forall s \in (0,\tau)\}.$$

Under the decreasing assumption on ϕ, one can identify the solutions to (C_e) with the curves $\left\{(x,v) \in [0,1] \times \mathbb{R} \mid \frac{v^2}{2} - \frac{1}{\mu}\phi(x) = \mathcal{E}\right\}$ for $\mathcal{E} \geqslant 0$. The phase-space $[0,1] \times \mathbb{R}$ is then splitted into two subdomains which are separated by the characteristic curve of equation

$$v = -\sqrt{\frac{2}{\mu}\left(\phi(x) - \phi_w\right)}.$$

One has the decomposition $[0,1] \times \mathbb{R} = D_1 \cup D_2$ with $D_1 := \{(x,v) \in [0,1] \times \mathbb{R} \mid v \geqslant -\sqrt{\frac{2}{\mu}\left(\phi(x) - \phi_w\right)}\}$ and $D_2 := \{(x,v) \in [0,1] \times \mathbb{R} \mid v < -\sqrt{\frac{2}{\mu}\left(\phi(x) - \phi_w\right)}\}$. For $(x,v) \in D_1$ there exists $w \geqslant 0$ and a characteristic curve passing through (x,v) which originates from $(0,w)$. Conversely, for $(x,v) \in D_2$ there exists $w < 0$ and a characteristic curve passing through (x,v) which originates from $(1,w)$. A geometry of the characteristics is illustrated in Figure 2.2.

Construction of a solution for the electrons.

Because of the boundary conditions and the geometry of the characteristics, we shall consider weak solutions of the Vlasov equation. Being a weak solution do not require ϕ to belong to $W^{2,\infty}(0,1)$, in fact $W^{1,\infty}(0,1)$ is sufficient. However, whenever it is possible to define the characteristic curves an explicit solution is easily constructed using the fact that solutions to the Vlasov equation have to be constant along the characteristics.

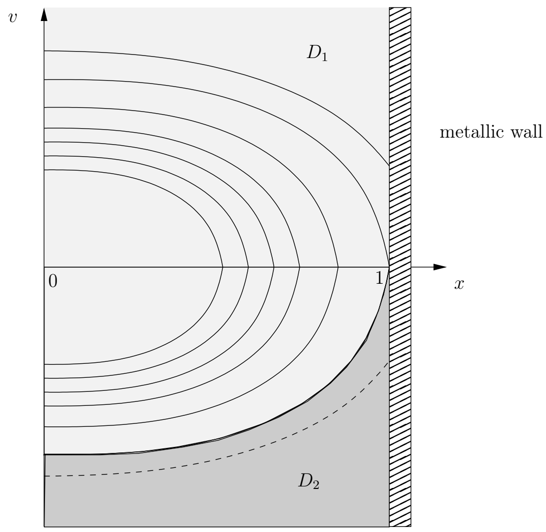


Figure 2.2 – Schematic electrons characteristic curves associated with a decreasing potential ϕ. The dashed line corresponds to a characteristic curve which originates at the wall with a negative velocity. Because of the boundary condition at the wall, particles following this curve were originally at $x = 0$ with a positive velocity.

Definition 2.3.3 (Weak solutions)**.** *Let $\alpha \in [0,1]$, $vf_e^{in} \in L^1_{loc}(\mathbb{R}^+)$ and $f_e \in L^1_{loc}([0,1] \times \mathbb{R})$. We say that f_e is a weak solution of the Vlasov problem for the electrons (2.13) iff for all $\varphi \in C^1_c([0,1] \times \mathbb{R})$ such that $\varphi(0,v) = 0$ for $v \leqslant 0$ and $\varphi(1,v) = -\alpha\varphi(1,-v)$ for $v \geqslant 0$*

$$-\int_0^1 \int_{\mathbb{R}} f_e(x,v)\Phi(x,v)dvdx = n_0 \int_{\mathbb{R}^+} f_e^{in}(v)v\varphi(0,v)dv$$

where Φ is the function defined by $\Phi(x,v) = v\partial_x\varphi(x,v) + \frac{1}{\mu}\frac{d}{dx}\phi(x)\partial_v\varphi(x,v)$.

Remark 1. *Note that in this definition, we only need $\frac{d}{dx}\phi \in L^\infty(0,1)$ to ensure the product $f_e\Phi$ to be integrable. However, with this regularity the characteristics are not well-defined.*

The solution for the linear Vlasov problem is not unique in general, it depends on the geometry of the characteristics. For instance, if there is a closed characteristic curve, namely a characteristic curve for which $t_{in} = -\infty$, then the distribution f_e can take arbitrary values on this curve. Nevertheless, with our assumption on the sign of ϕ' one can prove the uniqueness.

Theorem 2.3.4. *Let $\alpha \in [0,1]$ and $vf_e^{in} \in L^1_{loc}(\mathbb{R}^+)$ and $f_e \in L^1_{loc}([0,1] \times \mathbb{R})$ a weak solution of the Vlasov problem for the electrons. Then it is unique.*

Thanks to the partitioning of the phase space $[0,1] \times \mathbb{R} = D_1 \cup D_2$ and following the characteristics, we define the function f_e as follows:

$$f_e(x,v) = \begin{cases} n_0 f_e^{in}\left(\sqrt{v^2 - \frac{2}{\mu}\phi(x)}\right) & \text{if } (x,v) \in D_1 \\ \alpha n_0 f_e^{in}\left(\sqrt{v^2 - \frac{2}{\mu}\phi(x)}\right) & \text{if } (x,v) \in D_2. \end{cases} \tag{2.22}$$

Theorem 2.3.5. *Let $\alpha \in [0,1]$ and assume $f_e^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ then the function f_e defined by (2.22) is the weak solution of the Vlasov problem for the electrons.*

Even if f_e^{in} is a smooth function, the function f_e is not necessarily a classical solution of the Vlasov equation. Because of the geometry of the characteristics and the boundary conditions, f_e can be discontinus across the curves $S := \{(x,v) \in [0,1] \times \mathbb{R},\ v = -\sqrt{\frac{2}{\mu}(\phi(x) - \phi_w)}\}$. However, under some hypothesis on f_e^{in}, like for instance, that no particles are injected in a neighborhood of the velocity $v_c := \sqrt{-\frac{2}{\mu}\phi_w}$, it is possible to prove the following.

Proposition 2.3.6. *Let $\alpha \in [0,1)$ and $f_e^{in} \in C^1(\mathbb{R}^+)$. Assume moreover f_e^{in} vanishes in a neighborhood of v_c. Then f_e defined by (2.22) belongs to $C^1((0,1) \times \mathbb{R})$ and it is a classical solution of the Vlasov equation.*

Proof. It is clear from the definition (2.22) that f_e belongs to $C^1((0,1) \times \mathbb{R} \setminus S)$. Now since f_e^{in} vanishes in a neighborhood of v_c, f_e vanishes in a neighborhood of S and since for all $(x,v) \in S$, $f_e(x,v) = 0$, we therefore deduce that f_e is C^1 on $(0,1) \times \mathbb{R}$. We can therefore differentiate f_e and find that for all $(x,v) \in (0,1) \times \mathbb{R}$

$$v\partial_x f_e(x,v) + \frac{1}{\mu}\frac{d}{dx}\phi(x)\partial_v f_e(x,v) =$$

$$\begin{cases} n_0\left(-v\dfrac{\frac{d}{dx}\phi(x)(f_e^{in})'(\sqrt{v^2-\frac{2}{\mu}\phi(x)})}{\mu\sqrt{v^2-\frac{2}{\mu}\phi(x)}} + \dfrac{1}{\mu}\dfrac{d}{dx}\phi(x)\dfrac{(f_e^{in})'(\sqrt{v^2-\frac{2}{\mu}\phi(x)})}{\sqrt{v^2-\frac{2}{\mu}\phi(x)}}\right) = 0 \text{ if } (x,v) \in D_1, \\ \alpha n_0\left(-v\dfrac{\frac{d}{dx}\phi(x)(f_e^{in})'(\sqrt{v^2-\frac{2}{\mu}\phi(x)})}{\mu\sqrt{v^2-\frac{2}{\mu}\phi(x)}} + \dfrac{1}{\mu}\dfrac{d}{dx}\phi(x)\dfrac{(f_e^{in})'(\sqrt{v^2-\frac{2}{\mu}\phi(x)})}{\sqrt{v^2-\frac{2}{\mu}\phi(x)}}\right) = 0 \text{ if } (x,v) \in D_2. \end{cases}$$

Lastly, it easy to check that the boundary conditions are satisfied. □

First three moments of the electron distribution

It will be particularly important in the analysis of the full Vlasov-Poisson-Ampère system to have access to the density, the current density and the kinetic energy associated to the electron distribution f_e defined by (2.22).

Definition 2.3.7. *We define the density, the current density and the kinetic energy associated to the electron distribution f_e defined in (2.22) by the functions defined respectively for all $x \in [0,1]$ by*

$$n_e(x) := \int_{\mathbb{R}} f_e(x,v)dv, \quad \gamma_e(x) := \int_{\mathbb{R}} f_e(x,v)vdv, \quad \mathcal{E}_e(x) := \frac{1}{2}\int_{\mathbb{R}} f_e(x,v)v^2dv.$$

Proposition 2.3.8 (Electron density)**.** *Let $\alpha \in [0,1]$ and $f_e^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ then for all $x \in [0,1]$,*

$$n_e(x) = 2n_0 \int_{\sqrt{-\frac{2}{\mu}\phi(x)}}^{+\infty} \frac{f_e^{in}(v)v}{\sqrt{v^2 + \frac{2}{\mu}\phi(x)}}dv - n_0(1-\alpha)\int_{\sqrt{-\frac{2}{\mu}\phi_w}}^{+\infty} \frac{f_e^{in}(v)v}{\sqrt{v^2 + \frac{2}{\mu}\phi(x)}}dv \tag{2.23}$$

Proof. For $x \in [0,1]$ we split $\mathbb{R} = (-\infty, -\sqrt{\frac{2}{\mu}(\phi(x) - \phi_w)}) \cup [-\sqrt{\frac{2}{\mu}(\phi(x) - \phi_w)}, +\infty)$. Making the change of variable $w = \sqrt{v^2 - \frac{2}{\mu}\phi(x)}$ and integrating in velocity (2.22) leads to

$$n_e(x) = 2n_0 \int_{\sqrt{-\frac{2}{\mu}\phi(x)}}^{+\infty} \frac{f_e^{in}(w)w}{\sqrt{w^2 + \frac{2}{\mu}\phi(x)}} dw - n_0(1-\alpha) \int_{\sqrt{-\frac{2}{\mu}\phi_w}}^{+\infty} \frac{f_e^{in}(w)w}{\sqrt{w^2 + \frac{2}{\mu}\phi(x)}} dw.$$

□

Using a similar decomposition of the velocity line, we can give an expression for both the current and the kinetic energy.

Proposition 2.3.9 (Electron current density)**.** *Let $\alpha \in [0,1]$ and $f_e^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ such that $vf_e^{in} \in L^1(\mathbb{R}^+)$ then for all $x \in [0,1]$*

$$\gamma_e(x) = (1-\alpha)n_0 \int_{\sqrt{-\frac{2}{\mu}\phi_w}}^{+\infty} f_e^{in}(v)v dv. \tag{2.24}$$

Proposition 2.3.10 (Electron kinetic energy)**.** *Let $\alpha \in [0,1]$ and $f_e^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ such that $v^2 f_e^{in} \in L^1(\mathbb{R}^+)$ then for all $x \in [0,1]$*

$$\mathcal{E}_e(x) = n_0 \left(\int_{\sqrt{-\frac{2\phi(x)}{\mu}}}^{+\infty} f_e^{in}(w)w\sqrt{w^2 + \frac{2}{\mu}\phi(x)} dw - \frac{(1-\alpha)}{2} \int_{\sqrt{-\frac{2\phi_w}{\mu}}}^{+\infty} f_e^{in}(w)w\sqrt{w^2 + \frac{2}{\mu}\phi(x)} dw \right). \tag{2.25}$$

Study of ions characteristics

Definition 2.3.11. *The characteristics trajectories of ions (2.14) are the curves which satisfy the ordinary differential system of equations*

$$(C_i) : \begin{cases} \dot{\mathcal{X}}(t) = \mathcal{V}(t) \\ \dot{\mathcal{V}}(t) = -\frac{d}{dx}\phi(\mathcal{X}(t)) \\ \mathcal{X}(0) = x \\ \mathcal{V}(0) = v \end{cases}$$

for an arbitrary initial data $(x,v) \in \mathbb{R} \times [0,1] \times \mathbb{R}$. For any arbitrary initial data $(x,v) \in (0,1) \times \mathbb{R} \cup (\{0\} \times [0,+\infty)) \cup (\{1\} \times (-\infty,0])$ there is a unique solution denoted $(\mathcal{X}(t;x,v), \mathcal{V}(t;x,v))$ for all $t \in [t_{in}(x,v), t_{out}(x,v)]$ where

$$t_{in}(x,v) := \inf\{\tau \leqslant 0 : \mathcal{X}(s,x,v) \in (0,1), \forall s \in (\tau,0)\},$$

$$t_{out}(x,v) := \sup\{\tau \geqslant 0, \mathcal{X}(s,x,v) \in (0,1), \forall s \in (0,\tau)\}.$$

Again, under the decreasing assumption on ϕ the solutions to (C_i) can be identified with the curves $\left\{(x,v) \in [0,1] \times \mathbb{R} \,|\, \frac{v^2}{2} + \phi(x) = \mathcal{E}\right\}$ for $\mathcal{E} \geqslant \phi_w$. The phase space $[0,1] \times \mathbb{R}$ is then splitted into two subdomains which are separated by the characteristic curve of equation $v = \sqrt{-2\phi(x)}$. One has the decomposition $[0,1] \times \mathbb{R} = D_3 \cup D_4$ with $D_3 := \{(x,v) \in [0,1] \times \mathbb{R} \,|\, v > \sqrt{-2\phi(x)}\}$ and $D_4 := \{(x,v) \in [0,1] \times \mathbb{R} \,|\, v \leqslant \sqrt{-2\phi(x)}\}$. For $(x,v) \in D_3$ there exists $w > 0$ and a characteristic curve which originates from $(0,w)$. Conversely for $(x,v) \in D_4$ there exists $w \leqslant 0$ and a characteristic curve passing through (x,v) which originates from $(1,w)$. A geometry of the characteristics is illustrated in Figure 2.3.

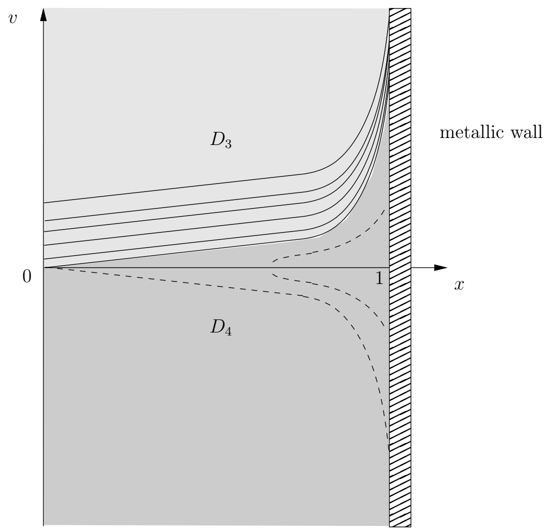


Figure 2.3 – Schematic ions characteristic curves associated with a decreasing potential ϕ. Here the dashed lines correspond to characteristic curves originating from the wall with negative velocities, and they span the darker gray region D_4. Because of the boundary condition at the wall, no particles travel on these curves and hence f_i vanishes outside D_3 (lighter gray).

Construction of a solution for the ions.

Because of the boundary conditions and the geometry of the characteristics, we shall consider weak solutions of the Vlasov equation.

Definition 2.3.12 (Weak solutions)**.** *Let $vf_i^{in} \in L^1_{loc}(\mathbb{R}^+)$ and $f_i \in L^1_{loc}([0,1] \times \mathbb{R})$. We say that f_i is a weak solution of the Vlasov problem for ions (2.14) iff for all $\varphi \in C^1_c([0,1] \times \mathbb{R})$ such that $\varphi(0,v) = 0$ for $v \leqslant 0$ and $\varphi(1,v) = 0$ for $v \geqslant 0$*

$$-\int_0^1 \int_{\mathbb{R}} f_i(x,v)\Phi(x,v)dvdx = \int_{\mathbb{R}^+} f_i^{in}(v)v\varphi(0,v)dv$$

where Φ is defined by $\Phi(x,v) = v\partial_x\varphi(x,v) - \frac{d}{dx}\phi(x)\partial_v\varphi(x,v)$.

As previously, thanks to the assumption on the sign of ϕ' we have uniqueness of the solution.

Theorem 2.3.13. *Let $vf_i^{in} \in L^1_{loc}(\mathbb{R}^+)$ and $f_i \in L^1_{loc}([0,1] \times \mathbb{R})$ a weak solution of the Vlasov problem for the ions. Then it is unique.*

Thanks to the partitioning of the phase space $[0,1] \times \mathbb{R} = D_3 \cup D_4$ and following the characteristics, we define the function f_i as follows:

$$f_i(x,v) = \begin{cases} f_i^{in}(\sqrt{v^2 + 2\phi(x)}) \text{ if } (x,v) \in D_3 \\ 0 \text{ if } (x,v) \in D_4. \end{cases} \tag{2.26}$$

One can then prove the following.

Theorem 2.3.14. *Let $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ then the function f_i defined by (2.26) is a weak solution of the Vlasov problem for the ion.*

As for the ions, we can also give a sufficient condition on the boundary f_i^{in} so that the solution f_i defined by (2.26) is in fact a classical one.

Proposition 2.3.15. *Let $f_i^{in} \in C^1(\mathbb{R}^+)$ be such that f_i^{in} vanishes in a neighborhood of zero. Then f_i defined by (2.26) belongs to $C^1((0,1) \times \mathbb{R})$ and it is a classical solution of the Vlasov equation.*

First three moments of the ion distribution

It will be particularly important in the analysis of the full Vlasov-Poisson-Ampère system to have access to the density, the current density and the kinetic energy associated to the ion distribution f_i defined by (2.26).

Definition 2.3.16. *We define the density, the current density and the kinetic energy associated to the ion distribution f_i defined in (2.26) by the functions defined respectively for all $x \in [0,1]$ by*

$$n_i(x) := \int_{\mathbb{R}} f_i(x,v)dv, \quad \gamma_i(x) := \int_{\mathbb{R}} f_i(x,v)vdv, \quad \mathcal{E}_i(x) := \frac{1}{2}\int_{\mathbb{R}} f_i(x,v)v^2dv.$$

Proposition 2.3.17 (Ion density)**.** *Let $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ then for all $x \in [0,1]$,*

$$n_i(x) = \int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{\sqrt{v^2 - 2\phi(x)}}dv. \tag{2.27}$$

Proposition 2.3.18 (Ion current density)**.** *Let $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ such that $vf_i^{in} \in L^1(\mathbb{R}^+)$ then for all $x \in [0,1]$,*

$$\gamma_i(x) = \int_{\mathbb{R}^+} f_i^{in}(v)vdv \tag{2.28}$$

Proposition 2.3.19 (Ion kinetic energy)**.** *Let $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ such that $v^2 f_i^{in} \in L^1(\mathbb{R}^+)$ then for all $x \in [0,1]$,*

$$\mathcal{E}_i(x) = \frac{1}{2}\int_{\mathbb{R}^+} f_i^{in}(v)v\sqrt{v^2 - 2\phi(x)}dv \tag{2.29}$$

Remark 2. *It is easy to see there is a little change in the geometry of the characteristics when ϕ' is permitted to vanish on some interval. However, when ϕ is not non increasing, it is possible that the characteristics curves are closed and never intersect the boundaries. This can lead to the presence of trapping sets of non zero-measure (see [2] for a definition of trapping sets) which results in the solution of (2.13)-(2.20) being non-unique.*

2.3.2 Maxwellian incoming electron boundary condition

Since we are interested to describe the transition between the plasma and the wall, we shall consider a Maxwellian boundary conditions for the electrons. Indeed, as mentionned in [63] electrons in the core of the plasma are well described by a full Maxwellian distribution as a matter of fact, the boundary conditions is taken of the form

$$F_e^{in}(V) := \sqrt{\frac{2m_e}{\pi k_B T_e}} e^{-\frac{m_e V^2}{2k_B T_e}} \quad \text{for } V > 0. \tag{2.30}$$

It gives in terms of dimensionless variables

$$f_e^{in}(v) := \sqrt{\frac{2\mu}{\pi}} e^{-\frac{\mu v^2}{2}} \quad \text{for } v > 0 \tag{2.31}$$

and note that $\int_{\mathbb{R}^+} f_e^{in}(v)dv = 1$. The electron density (2.23) is then given for all $x \in [0,1]$ by

$$n_e(x) = \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{\phi(x)} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} v}{\sqrt{v^2 + 2\phi(x)}} dv \right). \tag{2.32}$$

Notice that the electron density is close to a Boltzmannian density but not equal. It contains a perturbation that represents the truncation of the Maxwellian distribution due to the electron loss at the wall. The classical Boltzmannian density corresponds to the case $\alpha = 1$. The electron flux (2.24) is constant in space and given for all $x \in [0,1]$ by

$$\gamma_e(x) = (1-\alpha)\sqrt{\frac{2}{\pi\mu}} n_0 e^{\phi_w}. \tag{2.33}$$

The kinetic electron energy (2.25) is given for all $x \in [0,1]$ by

$$\mathcal{E}_e(x) = \frac{n_0}{\mu} \left(e^{\phi(x)} - \frac{(1-\alpha)}{\sqrt{2\pi}} \int_{\sqrt{-2\phi_w}}^{+\infty} e^{\frac{-v^2}{2}} v \sqrt{v^2 + 2\phi(x)} dv \right). \tag{2.34}$$

2.4 Determination of the wall potential ϕ_w and the reference plasma density n_0

In this section, we assume there is $n_0 > 0$, $\phi_w < 0$, $\phi \in W^{2,\infty}(0,1)$ with $\phi' < 0$, f_i and f_e that solves the Vlasov-Poisson-Ampère system (VPA) where $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$ with $v f_i^{in} \in L^1(\mathbb{R}^+)$ and f_e^{in} is the semi-Maxwellian distribution given by (2.31). We are going to show that thanks to the decreasing assumption on ϕ and the linear theory of section 2.3.1, it is possible to decouple the Ampère equation (2.16) and the charge imbalance relation (2.17) from the full (VPA) system. More precisely, we show that we can determine the couple (n_0, ϕ_w) so that these the two equations

$$\begin{cases} \forall x \in [0,1]\ \gamma_i(x) = \gamma_e(x) \\ (n_i - n_e)(0) = \rho_0 \end{cases}$$

are solved independently of ϕ.

2.4.1 The Ampère equation

Generally speaking, the potential at the wall cannot be a priori specified as a physical parameter. Therefore it is important to understand how it is determined in this model from other physical parameters. As mentioned in the introduction, the wall potential adjusts itself so that equal numbers of ions and electrons reach the wall per second. Following the idea in [63] Section 2.6 page 79, its value is determined from the ambipolarity. Thanks to the decreasing assumption on ϕ we have seen in section 2.3.1 that the ion flux and electron flux are given respectively for all $x \in [0,1]$ by

$$\gamma_i(x) = \int_{\mathbb{R}^+} f_i^{in}(v) v dv, \quad \gamma_e(x) = (1-\alpha)\sqrt{\frac{2}{\pi\mu}} n_0 e^{\phi_w} \ \forall \alpha \in [0,1],$$

so that the Ampère equation in fact reads for all $\alpha \in [0,1]$

$$\int_{\mathbb{R}^+} f_i^{in}(v) v dv = (1-\alpha)\sqrt{\frac{2}{\pi\mu}} n_0 e^{\phi_w}. \tag{2.35}$$

Remark 3. *We see that for $\alpha = 1$, the above equation implies $\int_{\mathbb{R}^+} f_i^{in}(v) v dv = 0$ and thus $f_i^{in}(v) = 0$ for a.e $v \in \mathbb{R}^+$. The model is therefore of no physical interest and it shows that the absorbing properties of the wall play a significant role in the establishement of the sheath.*

2.4.2 The charge imbalance relation

By definition the charge imbalance at $x = 0$ writes

$$\rho_0 := n_i(0) - n_e(0) = \int_{\mathbb{R}^+} f_i^{in}(v) dv - \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv \right). \tag{2.36}$$

We then observe that n_0 can be expressed as

$$n_0 = \sqrt{\frac{\pi}{2}} \frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) dv - \rho_0}{\left(\sqrt{2\pi} - (1-\alpha) \displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv \right)}. \tag{2.37}$$

Remark 4. *In the mathematical analysis Section 2.6, we will study the well-posedness of the above problems and consider ρ_0 as a given parameter. The value of n_0 will then be* defined *by (2.37). In order that n_0 be positive we observe that ρ_0 and f_i^{in} must be chosen such that $\rho_0 < \displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) dv$. Also in the next section, we will show that ϕ_w only depends on ρ_0, α and f_i^{in} and hence, so does n_0.*

2.4.3 The non linear equation on the floating potential

Substituting the expression (2.37) of n_0 in (2.35) leads for $\rho_0 \in \mathbb{R}$ and $\alpha \in [0,1)$ to the equivalent non linear relation

$$\mathcal{W}(\phi_w) = b \tag{2.38}$$

where $\mathcal{W} : \mathbb{R}^- \to \mathbb{R}$ is defined by

$$\mathcal{W}(\psi) := \frac{1}{\sqrt{\mu}} e^{\psi} \left(\int_{\mathbb{R}^+} f_i^{in}(v) dv - \rho_0 \right) + \int_{\mathbb{R}^+} f_i^{in}(v) v dv \int_{\sqrt{-2\psi}}^{+\infty} e^{-\frac{v^2}{2}} dv \tag{2.39}$$

and $b = \dfrac{\sqrt{2\pi}}{1-\alpha} \displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv$. Using standard arguments one has the following

Proposition 2.4.1 (Existence and uniqueness of the floating potential)**.** *Let $\rho_0 \in \mathbb{R}$ and $\alpha \in [0,1)$. Assume moreover that $f_i^{in} \in L^1(\mathbb{R}^+;\mathbb{R}^+)$ such that $v f_i^{in} \in L^1(\mathbb{R}^+)$ with $\int_{\mathbb{R}^+} f_i^{in}(v) dv > \rho_0$. Then the* *equation (2.38) has a unique non positive solution ϕ_w if and only if*

$$0 < \frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0} \leqslant \frac{(1-\alpha)}{(1+\alpha)}\sqrt{\frac{2}{\mu\pi}}. \tag{2.40}$$

Moreover the solution is in the interval,

$$\ln\left(\frac{\sqrt{2\pi}}{(1-\alpha)\left(\sqrt{2}+\frac{1}{\sqrt{\mu}}\left(\frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv}\right)\right)}\right) \leqslant \phi_w \leqslant 0. \tag{2.41}$$

Proof. Since $\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0 > 0$, the function $\mathcal{W}$ is continuous and increasing with $\displaystyle\lim_{-\infty}\mathcal{W} = 0$. Consequently, $\mathcal{W}$ is a bijection from $(-\infty, 0]$ to $(0, \mathcal{W}(0)]$ and the equation (2.38) admits a unique solution if and only if $b = \dfrac{\sqrt{2\pi}}{1-\alpha}\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv \in (0, \mathcal{W}(0)]$ which leads to the inequality (2.40). Now we prove the bounds (2.41). The upper bound is straightforward from the definition of the domain of $\mathcal{W}$. For the lower bound, after a change of variable in $\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}}dv$ we obtain

$$\mathcal{W}(\phi_w) = \frac{1}{\sqrt{\mu}}e^{\phi_w}\left(\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0\right) + \sqrt{2}\int_{\mathbb{R}^+} f_i^{in}(v)vdv \int_{\sqrt{-\phi_w}}^{+\infty} e^{-v^2}dv.$$

Then using the inequality $\displaystyle\int_{\sqrt{-\phi_w}}^{+\infty} e^{-u^2}du \leqslant \frac{e^{\phi_w}}{\sqrt{-\phi_w}+1}$ (see [52] page 163) we obtain

$$\mathcal{W}(\phi_w) \leqslant e^{\phi_w}\left(\frac{1}{\sqrt{\mu}}\left(\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0\right) + \frac{\sqrt{2}\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv}{\sqrt{-\phi_w}+1}\right).$$

Since $\sqrt{-\phi_w} + 1 \geqslant 1$, a simpler computable bound is then given by

$$\mathcal{W}(\phi_w) \leqslant e^{\phi_w}\left(\frac{1}{\sqrt{\mu}}\left(\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0\right) + \sqrt{2}\int_{\mathbb{R}^+} f_i^{in}(v)vdv\right)$$

and we conclude using the equality $\mathcal{W}(\phi_w) = \dfrac{\sqrt{2\pi}}{1-\alpha}\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv$. □

Remark 5. *The case of equality* $\dfrac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)vdv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0} = \dfrac{(1-\alpha)}{(1+\alpha)}\sqrt{\frac{2}{\mu\pi}}$ *yields the solution $\phi_w = 0$. However,* *this solution seems unphysical since in the sheath, the potential decreases.*

2.5 The non linear Poisson equation and its variational formulation

We remember that (NLP) is equivalent to the Vlasov-Poisson system when the electrostatic potential is decreasing, see Proposition 4.3.1. In the following section we will study the well posedness of (NLP) in the case of an incoming Maxwellian electron distribution. To this end, we will consider that $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and f_i^{in} satisfying (2.40) are given parameters as well as the normalized Debye length ε. The wall potential $\phi_w \leqslant 0$ will then be the solution of (2.38) and the electron boundary condition will be of the form (2.31) where the reference density n_0 is defined by (2.37). In particular, the (NLP) problem reformulates as follows : Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, f_i^{in} satisfying (2.40) and $\varepsilon > 0$. Find $\phi : [0,1] \to \mathbb{R}$ solution of

$$\text{(NLP-M)} \ : \begin{cases} -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = (n_i - n_e)(x) \text{ for all } x \in (0,1) \\ \text{with Dirichlet boundary conditions} \\ \phi(0) = 0 \text{ and } \phi(1) = \phi_w \\ \text{where} \\ n_i(x) = \displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{\sqrt{v^2 - 2\phi(x)}}dv, \\ n_e(x) = \dfrac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi}e^{\phi(x)} \\ \qquad - \dfrac{2n_0}{\sqrt{2\pi}}(1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}v}{\sqrt{v^2+2\phi(x)}}dv. \end{cases} \tag{2.42}$$

From a mathematical point of view, one would notice the analogy between the non linear Poisson equation and classical motion equations of a single particle in a potential force field. Indeed, the opposite of the right hand side of (4.51) derives from an abstract potential function $\mathcal{U} : [\phi_w, 0] \to \mathbb{R}$ given by

$$\mathcal{U}(\psi) := \int_{\mathbb{R}^+} f_i^{in}(v)v\sqrt{v^2 - 2\psi}dv + \frac{2n_0}{\sqrt{2\pi}}\left(\sqrt{2\pi}e^{\psi} - (1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}}v\sqrt{v^2+2\psi}dv\right). \tag{2.43}$$

The non linear Poisson equation (4.51) rewrites in the form

$$-\varepsilon^2\frac{d^2}{dx^2}\phi(x) = -\mathcal{U}'(\phi(x)) \quad \forall x \in (0,1), \tag{2.44}$$

indeed for all $x \in [0,1]$ $-\mathcal{U}'(\phi(x)) = (n_i - n_e)(x)$. Moreover, it can eventually be re-written into a variational form. Indeed, solutions to (4.51) are critical point of the energy functional

$$J_\varepsilon(\phi) := \int_0^1 \left(\frac{\varepsilon^2}{2}|\phi'(x)|^2 + \mathcal{U}(\phi(x))\right)dx \tag{2.45}$$

defined on the adequate functional space. Namely critical points of J_ε are solutions of $dJ_\varepsilon(\phi) \equiv 0$ where dJ_ε denotes the Fréchet derivative of J_ε. Therefore variational techniques are a convenient mathematical tool to solve the non linear Poisson equation. Historically speaking, variational methods to treat stationary transport problems were used in [46] to deal with neutron diffusion problems such as the Milne problem. It is also reminiscent of the work of [37] where variational techniques were also used to construct equilibria for a Vlasov-Poisson system in a polygonal domain.

2.6 Mathematical study of the non linear Poisson equation

In this section, we study the wellposedness of the non linear Poisson problem (NLP-M) which is equivalent to the Vlasov-Poisson-Ampère problem in the case where the incoming electron distribution is Maxwellian and the electrostatic potential is non increasing. We will use variational principles and the theory of Nemytskii's operator to study the functional J_ε formally defined by (2.45). Some theoretical results on Nemytskii's operator are reminded in the appendix. Let us define for all $\alpha \in [0,1)$

$$s_1(\alpha) := \frac{(1-\alpha)}{(1+\alpha)}\sqrt{\frac{2}{\mu\pi}}, \tag{2.46}$$

which is the upper bound in (2.40). We have a preliminary result that excludes non-interesting cases of the non linear Poisson equation.

Proposition 2.6.1. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $\varepsilon > 0$. Assume $f_i^{in} \in L^1(\mathbb{R}^+)$ with $vf_i^{in} \in L^1(\mathbb{R}^+)$ and such that the upper bound in (2.40) is an equality. Then*

1. *If $\rho_0 = 0$, $\phi \equiv 0$ is the unique non increasing solution of (NLP-M).*
2. *If $\rho_0 \neq 0$, there is no non increasing solution of (NLP-M).*

Proof. From proposition 2.4.1 we know that if $\frac{\int_{\mathbb{R}^+} f_i^{in}(v)vdv}{\int_{\mathbb{R}^+} f_i^{in}(v)dv - \rho_0} = s_1(\alpha)$ then $\phi_w = 0$. Consequently, if ϕ is a non increasing solution of (NLP-M) then $\phi_w \leqslant \phi \leqslant 0$ and so $\phi \equiv 0$. Then the non linear Poisson equation $-\varepsilon^2 \frac{d^2}{dx^2}\phi(x) = -\mathcal{U}'(\phi(x))$ for all $x \in (0,1)$ is equivalent to $\rho_0 = 0$ and the conclusion follows. $\square$

The physically relevant case corresponds to $\phi_w < 0$, so we shall only consider distributions f_i^{in} satisfying (2.40) with a strict inequality. Let us therefore define the functional framework that will be used in the following. In this work, we shall consider ion boundary conditions that are essentially bounded, integrable and of finite kinetic energy so that the potential $\mathcal{U}$ is well defined. However most of the results we will present can be extended without difficulty to the case where the incoming distribution f_i^{in} is a Dirac measure. We denote the set of such functions

$$\mathcal{I} := \left\{ h \in \left(L^1 \cap L^\infty\right)(\mathbb{R}^+;\mathbb{R}^+) \text{ such that } \int_{\mathbb{R}^+} h(v)v^2 dv < +\infty \right\}. \tag{2.47}$$

For $\rho_0 \in \mathbb{R}$ and $\alpha \in [0,1)$ given, we define the set of admissible ion boundary conditions

$$\mathcal{I}_{ad}(\rho_0,\alpha) := \left\{ h \in \mathcal{I} \text{ such that } 0 < \frac{\displaystyle\int_{\mathbb{R}^+} h(v)vdv}{\displaystyle\int_{\mathbb{R}^+} h(v)dv - \rho_0} < s_1(\alpha) \right\}, \tag{2.48}$$

as well as the set of admissible potential

$$V := V(\rho_0,\alpha) = \{\phi \in V_0 \mid \phi_w \leqslant \phi \leqslant 0 \text{ with } \phi(1) = \phi_w\}, \tag{2.49}$$

where $V_0 := \{\phi \in H^1(0,1) \mid \phi(0) = 0\}$ is a Hilbert space equipped with the inner product $(\phi, \varphi)_{V_0} := \int_0^1 \phi'(x)\varphi'(x)dx$ for any $(\phi, \varphi) \in V_0 \times V_0$ and with the induced norm defined by $\|\phi\|_{V_0} = \sqrt{(\phi,\phi)_{V_0}} = \|\phi\|_{H_0^1}$ for all $\phi \in V_0$. We also denote H^{-1} the dual space of $H_0^1(0,1)$, and we remind that the norm on H^{-1} is defined by $\|\mathcal{L}\|_{H^{-1}} := \sup_{\varphi \in H_0^1, \varphi \neq 0} \frac{|\langle \mathcal{L}, \varphi\rangle|}{\|\varphi\|_{H_0^1}}$ for all $\mathcal{L} \in H^{-1}$.

Remark 6. *In the definition of V, there is no assumptions on the monotonicity of ϕ. It is not necessary for $\mathcal{U}$ to be well-defined, but it will be shown in Theorem 2.6.3 that the solution ϕ of (NLP-M) is non increasing.*

Since the mass ratio always satisfies $\mu < \dfrac{2}{\pi}$ we define a critical re-emission coefficient

$$\alpha_c := \frac{1 - \sqrt{\frac{\pi}{2}}\sqrt{\mu}}{1 + \sqrt{\frac{\pi}{2}}\sqrt{\mu}} \tag{2.50}$$

which is such that $s_1(\alpha_c) = 1$ and $0 < \alpha_c < 1$. Let us now define, what a sheath solution stands for in our context.

Definition 2.6.2. *(Sheath solutions) Let (f_i, f_e, ϕ) be a solution to the Vlasov-Poisson-Ampère system (2.13)-(2.16). We say that it is a sheath-type solution on $(x^*, 1]$ with $0 \leqslant x^* < 1$ if on that interval ϕ is decreasing and $n_i > n_e$, and if $n_i(x^*) = n_e(x^*)$.*

We are now in position to state our main result.

Theorem 2.6.3. *Let $\alpha \in [0, \alpha_c]$, $\rho_0 = 0$, $f_i^{in} \in \mathcal{I}_{ad}(0, \alpha)$ and $\varepsilon > 0$. Let ϕ_w be the unique solution of (2.38). Assume the kinetic Bohm criterion*

$$\frac{\displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) dv} < \frac{\left(\sqrt{2\pi} + (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv\right)}{\left(\sqrt{2\pi} - (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv\right)}. \tag{2.51}$$

Then the Vlasov-Poisson-Ampère system (2.13)-(2.16) is well-posed, with a Maxwellian incoming electron distribution defined by (2.31), (2.37). More precisely, there is a unique $\phi_\varepsilon \in V$ solution of (NLP-M). In addition,

1. *The densities f_e and f_i defined in (2.22) and (2.26) are weak solutions of the Vlasov equations.*
2. *There exists $x^* \in [0,1)$ such that $(f_e, f_i, \phi_\varepsilon)$ is a sheath-type solution on $(x^*, 1]$ in the sense of Definition 2.6.2.*
3. *At the wall the values of n_i, n_e, ϕ_w and the velocity distributions f_i, f_e do not depend on the normalized Debye length ε.*
4. *ϕ_ε is $C^2[0,1]$, concave, non-increasing and we have the quantitative estimates*

$$\|\phi_\varepsilon\|_{V_0} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) \quad \text{and} \quad \|n_i - n_e\|_{H^{-1}} = \mathcal{O}(\varepsilon). \tag{2.52}$$

The proof of this theorem is proved in Section 2.6.2. Let us make a list of general but somewhat useful remarks about this theorem.

Remark 7. *A sufficient condition for* (2.51) *is* $\int_{\mathbb{R}+} \frac{f_i^{in}(v)}{v^2} dv < \int_{\mathbb{R}+} f_i^{in}(v)dv$. *This inequality still rewrites in physical variables* $\int_{\mathbb{R}+} \frac{F_i^{in}(V)}{V^2} dV < \frac{1}{c_s^2} \int_{\mathbb{R}+} F_i^{in}(V)dV$. *It coincides with the standard kinetic Bohm criterion presented in [63, Section 2.4].*

Remark 8. *The re-emission coefficient* α_c *is said to be critical because when* $\alpha > \alpha_c$ *we are not able to prove the existence of admissible boundary condition satisfying the kinetic Bohm criterion* (2.51). *On the contrary when* $\alpha \leqslant \alpha_c$ *we are able to do so (see theorem 2.6.16 and corollary 2.6.17).*

Remark 9. *In practice* α_c *is close to* 1 *and it allows to consider a wide range of material, even those with a high re-emission coefficient. As an example, for a Deuterium plasma* $\frac{1}{\mu} = 3672$ *and the critical re-emission coefficient is* $\alpha_c \approx 0.95$.

Remark 10. *In the theorem we have considered* $\rho_0 = 0$ *which corresponds to the neutrality* $n_i(0) = n_e(0)$. *It is an usual assumption in the physics literature. In the case* $\rho_0 \neq 0$ *and* $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ *for some* $\alpha \in [0, 1)$ *and* $\varepsilon > 0$, *we are able to establish the existence of a non increasing minimizer for* J_ε *see theorem 2.6.9 and proposition 2.6.11. However, since* V *is a strict and closed convex subset of* V_0, *minimizers are not necessarily critical points.*

Remark 11. *The kinetic Bohm criterion* (2.51) *expresses the strict local convexity of* $\mathcal{U}$ *at* $\psi = 0$. *Moreover, we have seen that* $(n_i - n_e)(x) = -\mathcal{U}'(\phi(x))$. *It follows that* $n_i - n_e$ *is a function of the electrostatic potential and we can verify*

$$\frac{d}{d\phi}(n_i - n_e)(x = 0) = -\mathcal{U}''(\phi(0)).$$

In particular (2.51) *is equivalent to* $\frac{d}{d\phi}(n_i - n_e)(0) < 0$ *which is an usual sheath criterion [18, 57].*

Remark 12. *The kinetic Bohm criterion implies that* $v^{-2} f_i^{in} \in L^1(\mathbb{R}^+; \mathbb{R}^+)$. *This means there is essentially no ions with null velocity at* $x = 0$. *In such a configuration, minimizers of* J_ε *are concave and non increasing solutions of the non linear Poisson equation. Thus* f_e *and* f_i *defined in* (2.22) *and* (2.26) *are weak solutions of the Vlasov equations and the Vlasov-Poisson-Ampère system is well posed. The uniqueness for the Poisson equation is proven by a reduction to a first order differential equation.*

Remark 13. *In the limit* $\varepsilon \to 0$, *the estimates* (2.52) *are mathematical expression of the quasi-neutrality.*

2.6.1 The minimization formulation

From now, we use the variational formulation of (NLP-M) and study the following minimization problem.

$$\begin{cases} \text{Let } \rho_0 \in \mathbb{R},\ \alpha \in [0,1),\ f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha) \text{ and } \varepsilon > 0. \\ \text{Find } \phi_\varepsilon^* \in V \text{ such that} \\ \phi_\varepsilon^* = \arg\min_{\phi \in V} J_\varepsilon(\phi). \end{cases} \tag{2.53}$$

Our approach consists in building the solution of (NLP-M) by minimizing the functional J_ε on V. This minimization problem is constrained, in the sense that the solution has to belong to the convex set V (2.49). The convexity of the set V is expressed in terms of pointwise inequalities that minimizers must satisfy. The theory for such minimization problem is well established. In particular, one could have drawn a parallel with the obstacle problem [61] where the the potential ϕ corresponds to the displacement of a line and the functional energy is the gravitational energy. Let us remember that our functional is defined for all $\phi \in V$ by

$$J_\varepsilon(\phi) = \int_0^1 \left(\frac{\varepsilon^2}{2} |\phi'(x)|^2 + \mathcal{U}(\phi(x)) \right) dx$$

where the real valued function $\mathcal{U}$ is defined for all $\psi \in [\phi_w, 0]$ by

$$\mathcal{U}(\psi) := \int_{\mathbb{R}^+} f_i^{in}(v) v \sqrt{v^2 - 2\psi} dv + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{\psi} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v \sqrt{v^2 + 2\psi} dv \right).$$

As we aim to find critical points of J_ε we need to know the regularity of $\mathcal{U}$. One has the following result.

Lemma 2.6.4. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. Then $\mathcal{U}$ is positive on $[\phi_w, 0]$ and of class C^1 on $[\phi_w, 0]$.*

Proof. Let $\psi \in [\phi_w, 0]$. Making the change of variable $w := \sqrt{v^2 + 2\psi}$ leads to

$$\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v \sqrt{v^2 + 2\psi} dv = e^{\psi} \int_{\sqrt{-2(\phi_w - \psi)}}^{+\infty} e^{-\frac{w^2}{2}} w^2 dw$$

and $\int_{\sqrt{-2(\phi_w - \psi)}}^{+\infty} e^{-\frac{w^2}{2}} w^2 dw \leqslant \int_0^{+\infty} e^{-\frac{w^2}{2}} w^2 dw = \frac{\sqrt{2\pi}}{2}$. We obtain finally

$$\mathcal{U}(\psi) \geqslant \int_{\mathbb{R}^+} f_i^{in}(v) v^2 dv + n_0(1+\alpha) e^{\phi_w} > 0.$$

Now we prove the regularity. The function $\mathcal{U}$ can be decomposed for all $\psi \in [\phi_w, 0]$ as

$$\mathcal{U}(\psi) = \mathcal{U}_1(\psi) + \mathcal{U}_2(\psi) + \mathcal{U}_3(\psi)$$

with

$$\begin{aligned} \mathcal{U}_1(\psi) &= \int_{\mathbb{R}^+} f_i^{in}(v) v \sqrt{v^2 - 2\psi} dv \\ \mathcal{U}_2(\psi) &= 2n_0 e^{\psi}, \\ \mathcal{U}_3(\psi) &= -\frac{2n_0(1-\alpha)}{\sqrt{2\pi}} \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v \sqrt{v^2 + 2\psi} dv. \end{aligned}$$

The function $\mathcal{U}_2$ clearly belongs to $C^1[\phi_w, 0]$. To prove the $\mathcal{U}_1$ and $\mathcal{U}_3$ are also C^1, it suffices to apply the theorem of differentiation under the integral sign. To do so we remark that the function defined respectively by

$$\begin{cases} u_1(v,\psi) := f_i^{in}(v) v \sqrt{v^2 - 2\psi} \text{ for a.e } v \in \mathbb{R}^+ \text{ for all } \psi \in [\phi_w, 0] \\ u_3(v,\psi) := e^{-\frac{v^2}{2}} v \sqrt{v^2 + 2\psi} \text{ for all } v \geqslant \sqrt{-2\phi_w} \text{ and } \psi \in [\phi_w, 0]. \end{cases}$$

are such that for all $\psi \in [\phi_w, 0]$, the partial functions $u_1(., \psi)$ and $u_3(., \psi)$ are respectively in $L^1(\mathbb{R}^+)$ and

$L^1(\sqrt{-2\phi_w}, +\infty))$ thanks to the bounds (uniform in ψ)

$$|u_1(v,\psi)| \leqslant f_i^{in}(v)v\sqrt{v^2-2\phi_w}, \text{ for a.e } v \in \mathbb{R}^+, \qquad |u_3(v,\psi)| \leqslant e^{-\frac{v^2}{2}}v^2 \text{ for all } v \geqslant \sqrt{-2\phi_w},$$

Their partial derivatives are also integrable thanks to the bounds (uniform in ψ)

$$|\frac{\partial u_1}{\partial \psi}(v,\psi)| \leqslant f_i^{in}(v) \text{ for a.e } v \in \mathbb{R}^+, \qquad |\frac{\partial u_2}{\partial \psi}(v,\psi)| \leqslant \frac{e^{-\frac{v^2}{2}}v}{\sqrt{v^2+2\phi_w}} \text{for all } v > \sqrt{-2\phi_w}.$$

□

We now consider the Nemytskii operator associated with $\mathcal{U}$ which is denoted $T_{\mathcal{U}}$ and defined for all $\phi \in V$ by $T_{\mathcal{U}}(\phi)(x) := \mathcal{U}(\phi(x))$ for all $x \in [0,1]$. Notice that J_ε is made of strictly convex part

$$\phi \in V \mapsto E_\varepsilon(\phi) = \frac{\varepsilon^2}{2}\|\phi\|_{V_0}^2 \tag{2.54}$$

plus a perturbation that is not necessarily convex

$$\phi \in V \mapsto F(\phi) = \int_0^1 T_{\mathcal{U}}(\phi)(x)dx. \tag{2.55}$$

All the analysis is based on the properties of the perturbation (2.55). We have the following.

Proposition 2.6.5. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$.*
Then $T_{\mathcal{U}} : V \to C^0[0,1]$ is of class C^1. Its Fréchet derivative is given by

$$dT_{\mathcal{U}}(\phi)(h) = \mathcal{U}'(\phi)h \quad \forall (\phi, h) \in V \times V_0.$$

Moreover the perturbation (2.55) is compact as we prove in the following.

Proposition 2.6.6. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. Then $T_{\mathcal{U}} : V \to C^0[0,1]$ is compact.*

Proof. We endow $C^0[0,1]$ with the norm $\phi \mapsto \|\phi\|_\infty := \max_{x\in[0,1]}|\phi(x)|$. One has $T_{\mathcal{U}} = \tilde{T_{\mathcal{U}}} \circ i$ where $i : V \to C^0([0,1];[\phi_w, 0])$ is the Rellich compact embedding and $\tilde{T_{\mathcal{U}}}$ is the restriction to $C^0([0,1];[\phi_w,0])$ of $T_{\mathcal{U}}$. Since i is compact and $\tilde{T_{\mathcal{U}}}$ is continuous, we conclude that $T_{\mathcal{U}}$ is compact. □

A direct consequence of the above proposition which results from Lemma 2.9.2 is the following.

Proposition 2.6.7. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$. Then $T_{\mathcal{U}}$ is (sequentially) weakly lower semicontinuous.*

Lemma 2.6.8. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$. Then $J_\varepsilon : V \to \mathbb{R}$ is well-defined, of class C^1 and (sequentially) wealky lower semicontinuous. Its Fréchet derivative is given by*

$$dJ_\varepsilon(\phi)(h) = \int_0^1 \varepsilon^2\phi'(x)h'(x) + \mathcal{U}'(\phi(x))h(x)dx \quad \textit{for all} \quad (\phi,h) \in V \times V_0. \tag{2.56}$$

Proof. First notice that for all $\phi \in V$

$$J_\varepsilon(\phi) = E_\varepsilon(\phi) + F(\phi)$$

where E_ε and F were given in (2.54)-(2.55). From Proposition 2.6.5 we deduce that F is C^1 over V and since E_ε is also C^1 over V thus J_ε is. For the weak lower semicontinuity, we notice that E_ε is convex and continuous for the strong topology, consequently applying the Mazur lemma [24, p. 562] we deduce that E_ε is sequentially weakly lower semicontinuous. Applying Proposition 2.6.7 we also deduce that F is weakly lower semicontinuous and thus J_ε is. ☐

We are now able to solve the minimization problem.

Theorem 2.6.9 (Existence of minimizers)**.** *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$. There is $\phi_\varepsilon^* \in V$ such that $J_\varepsilon(\phi_\varepsilon^*) \leqslant J_\varepsilon(\phi)$ for all $\phi \in V$. Moreover, the following estimate holds*

$$\|\phi_\varepsilon^*\|_{V_0} = \mathcal{O}\left(\frac{1}{\varepsilon}\right). \tag{2.57}$$

Proof. We apply Theorem 2.9.1. By definition V_0 is a reflexive Banach space and V is a closed convex subset. By Lemma 2.6.8, J_ε is sequentially weakly lower semicontinuous and since $\mathcal{U}$ is positive (Lemma 2.6.4) for all $\phi \in V$ we have

$$\frac{\varepsilon^2}{2}\|\phi\|_{V_0}^2 \leqslant J_\varepsilon(\phi).$$

By comparison $J_\varepsilon(\phi) \to +\infty$ as $\|\phi\|_{V_0} \to +\infty$ and thus J_ε is coercive. Therefore, there is $\phi_\varepsilon^* \in V$ such that $J_\varepsilon(\phi_\varepsilon^*) \leqslant J(\phi)$ for all $\phi \in V$. Finally, taking
$x \in [0,1] \mapsto \phi(x) := x\phi_w$ which belongs to V we obtain

$$\|\phi_\varepsilon^*\|_{V_0} \leqslant \sqrt{\phi_w^2 + \frac{2}{\varepsilon^2}\int_0^1 \mathcal{U}(x\phi_w)dx} = \mathcal{O}\left(\frac{1}{\varepsilon}\right). \tag{2.58}$$

☐

This theorem states the existence of global minimizers but does not ensure they are criticial point of J_ε. Let us give more properties of minimizers that will be useful in the sequel.

Proposition 2.6.10 (First order condition)**.** *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$. Let $\phi^* := \phi_\varepsilon^* \in V$ be a minimizer of J_ε. Then the following variational inequality holds*

$$dJ_\varepsilon(\phi^*)(h) \geqslant 0 \quad \text{for all } h \in V_0 \text{ such that } \phi^* + h \in V. \tag{2.59}$$

Moreover, we have $\phi^* \in W^{2,\infty}(0,1) \cap C^1[0,1]$ and

$$-\varepsilon^2 \frac{d^2}{dx^2}\phi^*(x) = -\mathcal{U}'(\phi^*(x)) \quad \text{a.e in } \mathcal{O} := \{x \in (0,1) \mid \phi_w < \phi^*(x) < 0\}, \tag{2.60}$$

$$-\varepsilon^2 \frac{d^2}{dx^2}\phi^*(x) \leqslant -\mathcal{U}'(\phi^*(x)) \quad \text{a.e in } \mathcal{F}_1 := \{x \in (0,1) \mid \phi^*(x) = 0\}, \tag{2.61}$$

$$-\varepsilon^2 \frac{d^2}{dx^2}\phi^*(x) \geqslant -\mathcal{U}'(\phi^*(x)) \quad \text{a.e in } \mathcal{F}_2 := \{x \in (0,1) \mid \phi^*(x) = \phi_w\}. \tag{2.62}$$

Proof. Since ϕ^* is a minimizer it is straightforward from the C^1-regularity of J_ε that we have the variational inequality (2.59). Then the regularity property $\phi^* \in W^{2,\infty}(0,1) \cap C^1[0,1]$ is obtained following exactly the same ideas as in [61, p. 113]. The equality (2.60) is obtained as follows. Choose $h \in H_0^1(0,1)$ with $\mathrm{supp}(h) \subset \mathcal{O}$ then there is $|\tau|$ sufficiently small such that $\phi^* + \tau h \in V$ and $\tau dJ_\varepsilon(\phi^*)(h) \geqslant 0$ for both positive and negative τ. Then $dJ_\varepsilon(\phi^*) \equiv 0$ and the result then follows from (2.56) and the regularity property $\phi^* \in W^{2,\infty}(0,1)$. Inequalities (2.61) and (2.62) can also be obtained from the first order condition (2.59) using adequate test functions $h \in H_0^1(0,1)$ with a support included respectively in $\mathcal{F}_1$ and $\mathcal{F}_2$. □

Now we prove that minimizers are necessarily non increasing functions.

Proposition 2.6.11 (Non increasing property)**.** *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$ and $\varepsilon > 0$. The minimizers of J_ε are non increasing functions. More precisely, if $\phi \in V$ is a minimizer and $0 \leqslant x \leqslant y \leqslant 1$ then $\phi(y) \leqslant \phi(x)$.*

Proof. Let $\phi \in V$ a minimizer of J_ε. Based on Theorem 1.1 of [11], we observe that the monotone decreasing rearrangement $\hat{\phi} : [0,1] \to \mathbb{R}$ of ϕ (that is the unique non increasing function such that for all $t \in \mathbb{R}$ $\text{meas}\left(\{x \in (0,1) \mid \phi(x) \geqslant t\}\right) = \text{meas}\Big(\{x \in (0,1) \mid \hat{\phi}(x) \geqslant t\}\Big)$) belongs to V and satisfies $F(\hat{\phi}) = F(\phi)$ and $E_\varepsilon(\hat{\phi}) \leqslant E_\varepsilon(\phi)$. In particular, if ϕ is not non increasing then the previous inequality is strict and $J_\varepsilon(\hat{\phi}) < J_\varepsilon(\phi)$ which contradicts the minimality of ϕ. □

2.6.2 Well-posedness of the Vlasov-Poisson-Ampère problem (2.13)-(2.17) and the kinetic Bohm criterion

In this section we give sufficient conditions on the function $\mathcal{U}$ so that minimizers of J_ε are critical points. Eventually, we give and show how the kinetic Bohm criterion is sufficient to obtain existence and uniqueness of critical points.

Proposition 2.6.12. *Let $\rho_0 \leqslant 0$, $\alpha \in [0,1)$, $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$ and $\varepsilon > 0$.*
Let $\phi^* := \phi_\varepsilon^* \in V$ a minimizer of J_ε. Assume $\mathcal{U}'(\phi_w) \leqslant 0$. Then $\phi^* \in C^2[0,1]$ and is solution of (NLP-M).

Proof. Let $\phi^* \in V$ a minimizer of J_ε, we will show that (2.60) holds on the all interval $(0,1)$. To do so we note that since ϕ^* is continuous and non increasing (see Proposition 2.6.11) there is $0 \leqslant \delta < \delta' \leqslant 1$ such that $\mathcal{F}_1 = (0,\delta]$ and $\mathcal{F}_2 = [\delta',1)$ (where $\mathcal{F}_1$ and $\mathcal{F}_2$ are the sets of Proposition 2.6.10). If $\mathcal{F}_1$ is non-empty then $\frac{d}{dx^2}\phi^* \equiv 0$ and (2.61) implies $0 \leqslant -\mathcal{U}'(0)$. Since $\mathcal{U}'(0) = -\rho_0 \geqslant 0$ by hypothesis, it follows that necessarily $0 = -\mathcal{U}'(0)$. Hence, if $\rho_0 = 0$ then $-\varepsilon^2 \frac{d^2}{dx^2}\phi^* = -\mathcal{U}'(\phi^*)$ on $\mathcal{F}_1$, else if $\rho_0 < 0$ then $\mathcal{F}_1$ is empty. The same argument holds for $\mathcal{F}_2$ using (2.62). We deduce $-\varepsilon^2 \frac{d^2}{dx^2}\phi^*(x) = -\mathcal{U}'(\phi^*(x))$ for almost every $x \in (0,1)$. Since $x \mapsto \mathcal{U}'(\phi^*(x)) \in C[0,1]$ we deduce from the Poisson equation that $\phi^* \in C^2[0,1]$. □

Remark 14. *Unfortunately, when $\rho_0 > 0$ we are not able to conclude to that minimizers are solutions to (NLP-M), however we are able to describe the behavior of $\mathcal{U}$, see Section 2.6.3 and Theorem 2.6.22.*

The cornerstone of this work is the following inequality also called the kinetic Bohm criterion.

Theorem 2.6.13 (Monotonicity of $\mathcal{U}$)**.** *Let $\rho_0 \in \mathbb{R}^+$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$. If*

$$\rho_0 > \int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv - \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} + (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv \right) \tag{2.63}$$

then $\mathcal{U}$ is decreasing. If the inequality is large then $\mathcal{U}$ is non increasing.

Remark 15. *When $\rho_0 = 0$ the inequality (2.63) is nothing but the inequality (2.51) of Theorem 2.6.3.*

To prove this theorem we shall need the two following inequalities that are obtained by a convexity argument.

Lemma 2.6.14. *For all $\eta > 0$ and $t^* \in (-\frac{1}{2\eta}, +\infty)$ we have*

$$\frac{e^t}{\sqrt{1+2\eta t}} \geqslant \frac{e^{t^*}}{\sqrt{1+2\eta t^*}} + \frac{e^{t^*}(t-t^*)}{\sqrt{1+2\eta t^*}} - \frac{\eta e^{t^*}(t-t^*)}{(1+2\eta t^*)^{\frac{3}{2}}}, \quad \forall t \in (-\frac{1}{2\eta}, +\infty). \tag{2.64}$$

Proof. For all $\eta > 0$, the function $h : t \in (-\frac{1}{2\eta}, +\infty) \mapsto \frac{e^t}{\sqrt{1+2\eta t}}$ is convex over $(-\frac{1}{2\eta}, +\infty)$. Indeed one has $h''(t) = \frac{e^t}{(1+2\eta t)^{\frac{5}{2}}}\left(4\eta^2 t^2 + 4\eta(1-\eta)t + 3\eta^2 - 2\eta + 1\right)$. The polynomial $t \mapsto 4\eta^2t^2 + 4\eta(1-\eta)t + 3\eta^2 - 2\eta + 1$ has for discriminant $\Delta = -32\eta^4$, hence if $\eta > 0$ then $\Delta < 0$ and $h''(t) > 0$. The conclusion follows from

$$h(t) \geqslant h(t^*) + (t-t^*)h'(t^*) \text{ for all } t, t^* \in (-\frac{1}{2\eta}, +\infty).$$

□

Lemma 2.6.15. *For all $\eta > 0$ and $t^* \in (-\infty, \frac{1}{2\eta})$, , we have*

$$\frac{e^t}{\sqrt{1-2\eta t}} \geqslant \frac{e^{t^*}}{\sqrt{1-2\eta t^*}} + \frac{e^{t^*}(t-t^*)}{\sqrt{1-2\eta t^*}} + \frac{\eta e^{t^*}(t-t^*)}{(1-2\eta t^*)^{\frac{3}{2}}} \quad \forall t \in (-\infty, \frac{1}{2\eta}) \tag{2.65}$$

Proof. The proof is similar to the previous one. For all $\eta > 0$, the function $h : t \in (-\infty, \frac{1}{2\eta}) \mapsto \frac{e^t}{\sqrt{1-2\eta t}}$ is convex over $(-\infty, \frac{1}{2\eta})$. Indeed one has $h''(t) = \frac{e^t}{(1-2\eta t)^{\frac{5}{2}}}\left(4\eta^2t^2 - 4\eta(\eta+1)t + 3\eta^2 + 2\eta + 1\right)$.
The polynomial $t \mapsto 4\eta^2t^2 - 4\eta(\eta-1)t + 3\eta^2 + 2\eta + 1$ has for discriminant $\Delta = -32\eta^4$, hence if $\eta > 0$ then $\Delta < 0$ and $h''(t) > 0$. The conclusion follows from

$$h(t) \geqslant h(t^*) + (t-t^*)h'(t^*) \text{ for all } t, t^* \in (-\infty, \frac{1}{2\eta}).$$

□

Proof of theorem 2.6.13. It is convenient to make the change of variable $u := -\psi$ and to define the function $u \in [0, -\phi_w] \mapsto \tilde{\mathcal{U}}(u) := \mathcal{U}(-u)$. We have $\tilde{\mathcal{U}} \in C^1[0, -\phi_w]$ and for all $u \in [0, -\phi_w]$

$$\frac{d}{du}\tilde{\mathcal{U}}(u) = e^{-u}\left(\underbrace{\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^u}{\sqrt{1+\frac{2u}{v^2}}}dv}_{:=A} - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi} + \frac{2n_0}{\sqrt{2\pi}}\left((1-\alpha)\underbrace{\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^u}{\sqrt{1-\frac{2u}{v^2}}}dv}_{:=B}\right)\right).$$

We shall give a lower bound for A and B. Applying respectively inequalities (2.64) and (2.65) to the integrands of A and B with $u^* = 0$ and $\eta = \frac{1}{v^2}$, we obtain

$$\begin{aligned}\frac{d}{du}\tilde{\mathcal{U}}(u) \geqslant & e^{-u}\left(\int_{\mathbb{R}^+} f_i^{in}(v)\left(1 + u(1-\frac{1}{v^2})\right)dv - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi}\right) + \\ & e^{-u}\left(\frac{2n_0}{\sqrt{2\pi}}(1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}}\left(1 + u(1+\frac{1}{v^2})\right)dv\right)\end{aligned}$$

we therefore obtain

$$\frac{d}{du}\tilde{\mathcal{U}}(u) \geqslant e^{-u}\rho_0 + e^{-u}u\left[\rho_0 + \frac{2n_0}{\sqrt{2\pi}}\left(\sqrt{2\pi} + (1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty}\frac{e^{-\frac{v^2}{2}}}{v^2}dv\right) - \int_{\mathbb{R}^+}\frac{f_i^{in}(v)}{v^2}dv\right].$$

By hypothesis $\rho_0 \geqslant 0$ hence for all $u \in (0, -\phi_w]$ we have that $\frac{d}{du}\tilde{\mathcal{U}}(u)$ is positive if the bracket is positive and non negative if the bracket vanishes. □

To be completely self-consistent, we must now investigate the existence of admissible ion boundary conditions f_i^{in} that satisfy the kinetic Bohm criterion (2.51). This is at this stage of the analysis that the critical re-emission coefficient α_c comes up. Indeed, let us define for all $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(0,\alpha)$

$$s_2(\alpha) := \frac{\left(\sqrt{2\pi} + (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty}\frac{e^{-\frac{v^2}{2}}}{v^2}dv\right)}{\left(\sqrt{2\pi} - (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty}e^{-\frac{v^2}{2}}dv\right)}. \tag{2.66}$$

and notice that $s_2(\alpha) > 1$. We have the following characterization of existence result.

Theorem 2.6.16. *Let $\alpha \in [0,1)$. There exists $f_i^{in} \in \mathcal{I}_{ad}(0,\alpha)$ satisfying the kinetic Bohm criterion (2.51) if and only if $s_1(\alpha)^2 s_2(\alpha) > 1$.*

Proof. Let $\alpha \in [0,1)$. We begin with showing the necessary condition. Assume there exists $f_i^{in} \in \mathcal{I}$ satisfying

$$\frac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)v\,dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv} < s_1(\alpha) \quad \text{and} \quad \frac{\displaystyle\int_{\mathbb{R}^+}\frac{f_i^{in}(v)}{v^2}dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v)dv} < s_2(\alpha). \tag{2.67}$$

Applying twice the Cauchy Schwarz inequality yields

$$\int_{\mathbb{R}^+} f_i^{in}(v)dv \leqslant \left(\int_{\mathbb{R}^+} f_i^{in}(v)v\,dv\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^+}\frac{f_i^{in}(v)}{v^2}dv\right)^{\frac{1}{4}}\left(\int_{\mathbb{R}^+} f_i^{in}(v)dv\right)^{\frac{1}{4}}.$$

Using the previous inequalities (2.67) we obtain $1 < s_1(\alpha)^2 s_2(\alpha)$. Let us now prove the sufficient condition. Assume $s_1^2(\alpha)s_2(\alpha) > 1$. Then we claim that the function f_i^{in} defined for all $v \in \mathbb{R}^+$ by $f_i^{in}(v) = \mathbf{1}_{(v_{min},v_{max})}(v)$ with $v_{min} = \frac{1}{\sqrt{s_2(\alpha)}}$ and $v_{max} = s_1(\alpha)$ is a solution. □

A direct consequence of the previous result is the following.

Corollary 2.6.17. *Let $\alpha \in [0,\alpha_c]$. Then there exists $f_i^{in} \in \mathcal{I}_{ad}(0,\alpha)$ satisfying the kinetic Bohm criterion (2.51).*

Proof. For all $\alpha \in [0,\alpha_c]$, $s_1(\alpha) \geqslant 1$. Since $s_2(\alpha) > 1$ we deduce $s_1^2(\alpha)s_2(\alpha) > 1$ and Theorem 2.6.16 applies. □

Let us now prove the main result 2.6.3.

Proof of theorem 2.6.3. Let $\alpha \in [0, \alpha_c]$, $\rho_0 = 0$, $f_i^{in} \in \mathcal{I}_{ad}(0, \alpha)$ and $\varepsilon > 0$. Moreover, assume the kinetic Bohm criterion

$$\frac{\displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv}{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) v dv} < \frac{\left(\sqrt{2\pi} + (1-\alpha) \displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv\right)}{\left(\sqrt{2\pi} - (1-\alpha) \displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv\right)}.$$

The proof is splitted into two parts. Mainly, the first part deals with the existence of a solution the Vlasov-Poisson-Ampère system and the second part deals with the uniqueness.

Existence part We apply the proposition 2.6.13, so that the function $\phi \in [\phi_w, 0] \mapsto \mathcal{U}(\phi)$ is decreasing and $\mathcal{U}'(\phi) < 0$ for all $\phi \in [\phi_w, 0)$. Combining theorem 2.6.9 and corollary 2.6.12 we obtain there is $\phi_\varepsilon \in V \cap C^2[0,1]$ non increasing solution of (NLP-M). Since $\varepsilon^2 \frac{d}{dx^2}\phi_\varepsilon(x) = \mathcal{U}'(\phi_\varepsilon(x)) \leqslant 0$ for all $x \in (0,1)$ we deduce that ϕ_ε is concave on $[0,1]$. Now considering f_e and f_i defined in (2.22) and (2.26), it is easy to see that they are weak solution of the Vlasov equations. Now it easy to observe that $(f_i, f_e, \phi_\varepsilon)$ is a sheath type solution on $(x^*, 1]$ where $x^* = \max\{x \in [0,1] \,/\, \phi_\varepsilon(x) = 0\}$. Besides, the Ampère equation (2.9) is satisfied by definition of ϕ_w and we therefore deduce that the Vlasov-Poisson-Ampère system (2.13)-(2.16) is has a solution. In addition, it is straightforward from the equation (2.38) that ϕ_w does not depend on ε and so do $v \mapsto (f_i(1,v), f_e(1,v))$, $n_i(1)$ and $n_e(1)$. We shall now prove the estimates (2.52). The first one is obtained from (2.57). The second one is obtained as follows. Since $n_i - n_e$ is a continuous function and by the canonical injection $C^0[0,1] \hookrightarrow H^{-1}(0,1)$, it defines a linear and continuous form on the space $H_0^1(0,1)$ and we have for all $\psi \in H_0^1(0,1)$

$$\langle n_i - n_e, \psi \rangle_{H^{-1}, H_0^1} = \int_0^1 (n_i - n_e)(x)\psi(x) dx = \varepsilon^2 \int_0^1 \frac{d}{dx}\phi_\varepsilon(x) \frac{d}{dx}\psi(x) dx.$$

Using the Cauchy-Schwarz inequality and the estimate (2.58) we obtain

$$\left| \langle n_i - n_e, \psi \rangle_{H^{-1}, H_0^1} \right| \leqslant \varepsilon^2 \|\phi_\varepsilon\|_{V_0} \|\psi\|_{H_0^1} \leqslant \varepsilon^2 \sqrt{\phi_w^2 + \frac{2}{\varepsilon^2} F(x\phi_w)} \|\psi\|_{H_0^1}$$

which leads to $\|n_i - n_e\|_{H^{-1}} \leqslant \varepsilon^2 \sqrt{\phi_w^2 + \frac{2}{\varepsilon^2} F(x\phi_w)} = \mathcal{O}(\varepsilon)$.

Uniqueness part The proof of the uniqueness result relies on a reduction of the non linear Poisson equation to a first order differential equation. We shall also need the following lemma whose proof is a consequence of the Cauchy-Lipschitz theorem.

Lemma 2.6.18. *Let $0 \leqslant x_1 < x_2 \leqslant 1$ and $\phi \in C^1([x_1, x_2]; [\phi_w, 0])$ solution of the initial Cauchy problem*

$$\begin{cases} \frac{d}{dx}\phi(x) = -\sqrt{g(\phi(x))} \ \textit{where } g : [\phi_w, 0] \mapsto (0, +\infty) \ \textit{is } C^1[\phi_w, 0] \\ \phi(x_2) = \phi_2 \in \mathbb{R}. \end{cases}$$

then it is unique.

Proof. Since g is C^1 and $g > 0$, the function $\phi \in [\phi_w, 0] \mapsto -\sqrt{g(\phi)}$ is Lipschitz in ϕ and it suffices to apply the Cauchy-Lipschitz theorem to conclude. □

We are now able to prove the uniqueness result. To this effect, let us multiply the non linear Poisson equation (2.44) by $\frac{d}{dx}\phi_\varepsilon$ and integrate over an arbitrary segment $[x, y] \subset [0,1]$ with $0 \leqslant x < y \leqslant 1$. Then we obtain

$$\frac{\varepsilon^2}{2}\left(\left(\frac{d}{dx}\phi_\varepsilon(y)\right)^2-\left(\frac{d}{dx}\phi_\varepsilon(x)\right)^2\right)=\mathcal{U}(\phi_\varepsilon(y))-\mathcal{U}(\phi_\varepsilon(x)). \tag{2.68}$$

Further assume there is a solution $\psi_\varepsilon \in V\cap C^2[0,1]$ concave, non increasing and different of ϕ_ε. Since ϕ_ε and ψ_ε have the same boundary conditions and are continuous, there exist $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$, $\phi_\varepsilon < \psi_\varepsilon$ on (x_1,x_2), and $\phi_\varepsilon(x_1)=\psi_\varepsilon(x_1)$, $\phi_\varepsilon(x_2)=\psi_\varepsilon(x_2)<0$. Then $\frac{d}{dx}\phi_\varepsilon(x_1)\leqslant\frac{d}{dx}\psi_\varepsilon(x_1)\leqslant 0$ and $\frac{d}{dx}\psi_\varepsilon(x_2)\leqslant\frac{d}{dx}\phi_\varepsilon(x_2)<0$. Since the previous relation (2.68) is valid for any $0\leqslant x<y\leqslant 1$, chosing $x=x_1$ and $y=x_2$ leads to

$$\left(\frac{d}{dx}\psi_\varepsilon(x_2)\right)^2-\left(\frac{d}{dx}\psi_\varepsilon(x_1)\right)^2=\left(\frac{d}{dx}\phi_\varepsilon(x_2)\right)^2-\left(\frac{d}{dx}\phi_\varepsilon(x_1)\right)^2,$$

and by a comparison argument we obtain $\frac{d}{dx}\psi_\varepsilon(x_1)=\frac{d}{dx}\phi_\varepsilon(x_1)$ and $\frac{d}{dx}\phi_\varepsilon(x_2)=\frac{d}{dx}\psi_\varepsilon(x_2)$. Eventually using the relation (2.68) for $y=x_2$ and $x_1\leqslant x\leqslant x_2$ it is easy to notice that ϕ_ε and ψ_ε are both solutions of the Cauchy problem

$$\begin{cases}\dfrac{\varepsilon}{\sqrt{2}}\dfrac{d}{dx}w(x)=-\sqrt{\mathcal{U}(w(x))-\mathcal{U}(\phi_\varepsilon(x_2))+\dfrac{\varepsilon^2}{2}\dfrac{d}{dx}\phi_\varepsilon(x_2)^2}\ \text{for } x\geqslant x_1\\ w(x_1)=\phi_\varepsilon(x_1).\end{cases}$$

Notice that $\mathcal{U}(w(x))-\mathcal{U}(\phi_\varepsilon(x_2))+\frac{\varepsilon^2}{2}\frac{d}{dx}\phi_\varepsilon(x_2)^2=\mathcal{U}(w(x))-\mathcal{U}(\phi_\varepsilon(x_1))+\frac{\varepsilon^2}{2}\frac{d}{dx}\phi_\varepsilon(x_2)^2$ and also that $\mathcal{U}(w(x))\geqslant\mathcal{U}(\phi_\varepsilon(x_1))$ because $\mathcal{U}$ and w are non increasing. Finally remark that $\mathcal{U}(w(x))-\mathcal{U}(\phi_\varepsilon(x_1))+\frac{\varepsilon^2}{2}\frac{d}{dx}\phi_\varepsilon(x_2)^2\geqslant\frac{\varepsilon^2}{2}\frac{d}{dx}\phi_\varepsilon(x_2)^2>0$ and conclude by invoking lemma 2.6.18. □

The proof of uniqueness would be simplified if the function $\mathcal{U}$ were convex. So we give a sufficient condition on f_i^{in} that is stronger than the kinetic Bohm criterion (2.51) of Theorem 2.6.3.

Proposition 2.6.19. *Let $\rho_0=0$, $\alpha\in[0,1)$ and $f_i^{in}\in\mathcal{I}_{ad}(0,\alpha)$. Assume*

$$\frac{\displaystyle\int_{\mathbb{R}^+}\frac{f_i^{in}(v)}{v^2}dv}{\displaystyle\int_{\mathbb{R}^+}f_i^{in}(v)dv}\leqslant\inf_{\psi\in(\phi_w,0]}\mathcal{V}_e^{-2}(\psi),\ \textit{where}\ \mathcal{V}_e^{-2}(\psi)=\frac{\left(\sqrt{2\pi}e^{\psi}+(1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty}\frac{e^{-\frac{v^2}{2}}v}{(v^2+2\psi)^{\frac{3}{2}}}dv\right)}{\left(\sqrt{2\pi}-(1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty}e^{-\frac{v^2}{2}}dv\right)}. \tag{2.69}$$

Then $\mathcal{U}$ is convex.

Proof. Note that the function $\mathcal{V}_e^{-2}$ is positive and continuous on $(\phi_w,0]$ with $\lim_{\psi\to\phi_w^+}\mathcal{V}_e^{-2}(\psi)=+\infty$ and thus the infimum exists in $\mathbb{R}$ and it is reached at some points in $(\phi_w,0]$. Consequently the ion boundary condition is such that $v^{-2}f_i^{in}\in L^1(\mathbb{R}^+)$ and one can show that $\mathcal{U}$ belongs to $C^2(\phi_w,0]$. One has therefore for all $\psi\in(\phi_w,0]$,

$$\mathcal{U}''(\psi)=-\int_{\mathbb{R}^+}\frac{f_i^{in}(v)v}{(v^2-2\psi)^{\frac{3}{2}}}dv+\frac{2n_0}{\sqrt{2\pi}}\left(\sqrt{2\pi}e^{\psi}+(1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty}\frac{e^{-\frac{v^2}{2}}v}{(v^2+2\psi)^{\frac{3}{2}}}dv\right).$$

Now since for all $\psi \in (\phi_w, 0]$ and $v > 0$, $v^2 - 2\psi \geqslant v^2$ we deduce the inequality

$$\mathcal{U}''(\psi) \geqslant -\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{\psi} + (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}} v}{(v^2 + 2\psi)^{\frac{3}{2}}} dv \right).$$

Since $n_0 = \sqrt{\frac{\pi}{2}} \dfrac{\displaystyle\int_{\mathbb{R}^+} f_i^{in}(v) dv}{\left(\sqrt{2\pi} - (1-\alpha) \displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv \right)}$ we obtain

$$\mathcal{U}''(\psi) \geqslant -\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv + \int_{\mathbb{R}^+} f_i^{in}(v) dv \mathcal{V}_e^{-2}(\psi) \geqslant 0.$$

Since $\mathcal{U}''$ is non negative, we deduce that $\mathcal{U}$ is convex. □

Remark 16. *The difference between the inequality (2.69) and the kinetic Bohm criterion (2.51) resides in the fact that the upper bound in (2.51) is $\mathcal{V}_e^{-2}(0)$. But $\mathcal{V}_e^{-2}(0) \geqslant \inf_{\psi \in (\phi_w, 0]} \mathcal{V}_e^{-2}(\psi)$ and it is not so clear whether this inequality is an equality.*

2.6.3 Complementary study when the kinetic Bohm criterion is violated

In the theorem 2.6.3, we have assumed that $\rho_0 = 0$ and the kinetic Bohm criterion (2.51) holds true. In this section, we intend to consider more general cases, that is : either the kinetic Bohm criterion is violated or $\rho_0 \neq 0$. We remember that for $\rho_0 \in \mathbb{R}$, $\alpha \in [0, 1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ given, $\mathcal{U}$ is defined for all $\psi \in [\phi_w, 0]$ by

$$\mathcal{U}(\psi) := \int_{\mathbb{R}^+} f_i^{in}(v) v \sqrt{v^2 - 2\psi} dv + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{\psi} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v \sqrt{v^2 + 2\psi} dv \right).$$

We intend to prove that the general situation is that $\mathcal{U}$ admits at most two local minima and two local maxima provided it is not flat anywhere. As a consequence, it shows that whenever the Vlasov-Poisson-Ampère is well posed with a potential ϕ non increasing, the charge density $x \in [0, 1] \mapsto (n_i - n_e)(x) = -\mathcal{U}'(\phi(x))$ can change sign at most in three distinct regions. This section is thus devoted to the study of the monotonicity of $\mathcal{U}$. To do so, it is convenient to make the change of variable $u := -\psi$ and to define the function $u \in [0, -\phi_w] \mapsto \tilde{\mathcal{U}}(u) := \mathcal{U}(-u)$. We will assume in this section that $\displaystyle\int_{\mathbb{R}^+} \frac{f_i^{in}(v)}{v^2} dv < +\infty$ so that $\tilde{\mathcal{U}} \in C^1[0, -\phi_w] \cap C^2[0, -\phi_w)$ and for all $u \in [0, -\phi_w)$,

$$\tilde{\mathcal{U}}(u) = \int_{\mathbb{R}^+} f_i^{in}(v) v \sqrt{v^2 + 2u} dv + \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} e^{-u} - (1-\alpha) \int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} v \sqrt{v^2 - 2u} dv \right), \quad (2.70)$$

$$\tilde{\mathcal{U}}'(u) = \int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{\sqrt{v^2+2u}}dv$$
$$+ \frac{2n_0}{\sqrt{2\pi}} \left(-\sqrt{2\pi}e^{-u} + (1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}v}{\sqrt{v^2-2u}}dv \right), \quad (2.71)$$

$$\tilde{\mathcal{U}}''(u) = -\int_{\mathbb{R}^+} \frac{f_i^{in}(v)v}{(v^2+2u)^{\frac{3}{2}}}dv$$
$$+ \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi}e^{-u} + (1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}v}{(v^2-2u)^{\frac{3}{2}}}dv \right). \quad (2.72)$$

Proposition 2.6.20. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$. If $\tilde{\mathcal{U}}$ has a local minimum at $u^* \in (0,-\phi_w)$ then $\tilde{\mathcal{U}}$ is non decreasing over $[u^*,-\phi_w]$.*

Proof. Assume $\tilde{\mathcal{U}}$ attains a minimum at $u^* \in (0,-\phi_w)$ then one has the first and second order conditions $\frac{d}{du}\tilde{\mathcal{U}}(u^*) = 0$ and $\frac{d^2}{du^2}\tilde{\mathcal{U}}(u^*) \geqslant 0$ that is

$$\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv + \frac{2n_0}{\sqrt{2\pi}} \left((1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv - \sqrt{2\pi} \right) = 0 \quad (2.73)$$

and

$$\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{v^2\left(1+\frac{2u^*}{v^2}\right)^{\frac{3}{2}}}dv \leqslant \frac{2n_0}{\sqrt{2\pi}} \left(\sqrt{2\pi} + (1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{v^2\left(1-\frac{2u^*}{v^2}\right)^{\frac{3}{2}}}dv \right). \quad (2.74)$$

One has the decomposition for all $u \in [0,-\phi_w]$

$$\frac{d}{du}\tilde{\mathcal{U}}(u) = e^{-u}\left(\underbrace{\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^u}{\sqrt{1+\frac{2u}{v^2}}}dv}_{:=A} \right.$$
$$\left. + e^{-u}\left(-\frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi} + \frac{2n_0}{\sqrt{2\pi}}\left((1-\alpha)\underbrace{\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^u}{\sqrt{1-\frac{2u}{v^2}}}dv}_{:=B} \right)\right)\right).$$

Let us now give a lower bound for A. Applying the inequality (2.64) we have

$$A \geqslant \int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv + (u-u^*)\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv$$
$$-(u-u^*)\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{v^2\left(1+\frac{2u^*}{v^2}\right)^{\frac{3}{2}}}dv$$

then we use the second order condition (2.74) and obtain for all $u \geqslant u^*$

$$A \geqslant \int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv$$
$$+(u-u^*)\left(\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi} - \frac{2n_0}{\sqrt{2\pi}}(1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{v^2\left(1-\frac{2u^*}{v^2}\right)^{\frac{3}{2}}}dv\right).$$

We also have a lower bound for B. Indeed, using the inequality (2.65) we have

$$B \geqslant \int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv$$
$$+(u-u^*)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv + (u-u^*)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{v^2\left(1-\frac{2u^*}{v^2}\right)^{\frac{3}{2}}}dv.$$

Combining A and B one finally obtains

$$\frac{d}{du}\tilde{\mathcal{U}}(u) \geqslant e^{-u}\left(\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv + \frac{2n_0}{\sqrt{2\pi}}(1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi}\right)$$
$$+e^{-u}(u-u^*)\left(\int_{\mathbb{R}^+} \frac{f_i^{in}(v)e^{u^*}}{\sqrt{1+\frac{2u^*}{v^2}}}dv + \frac{2n_0}{\sqrt{2\pi}}(1-\alpha)\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}e^{u^*}}{\sqrt{1-\frac{2u^*}{v^2}}}dv - \frac{2n_0}{\sqrt{2\pi}}\sqrt{2\pi}\right)$$

and the right hand side is exactly zero so that $\frac{d}{du}\tilde{\mathcal{U}}(u) \geqslant 0$ for all $u \geqslant u^*$. □

From Proposition 2.6.20 one can establish the following result.

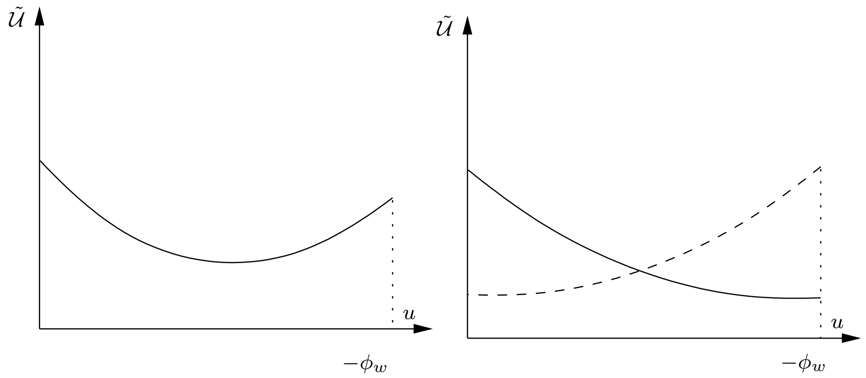
Proposition 2.6.21. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0,1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0,\alpha)$ such that $\tilde{\mathcal{U}}$ is not locally constant. If $\tilde{\mathcal{U}}$ attains a local minimum over $(0,-\phi_w)$ then it is unique. Similarly, if $\tilde{\mathcal{U}}$ attains a local maxima in $(0,-\phi_w)$ then it is unique.*

Proof. We do the proof by contradiction. Assume $\tilde{\mathcal{U}}$ has at least two local minima at some points u_1 and u_2 belonging to $(0,-\phi_w)$. Without loss of generality we can assume $u_1 < u_2$. Since $\tilde{\mathcal{U}}$ is not constant on (u_1,u_2) and $C^2[u_1,u_2]$ there is $u_1 < \delta < u_2$ such that $\tilde{\mathcal{U}}$ is decreasing over (δ,u_2) which is contradiction with Proposition 2.6.20. We can also prove by similar arguments that if $\tilde{\mathcal{U}}$ has a local maxima in $(0,-\phi_w)$ then it is unique. □

Theorem 2.6.22. *Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0, 1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ such that $\tilde{\mathcal{U}}$ is not locally constant. Then $\tilde{\mathcal{U}}$ admits at most two local minima over $[0, -\phi_w]$ and it also admits at most two local maxima over $[0, -\phi_w]$.*

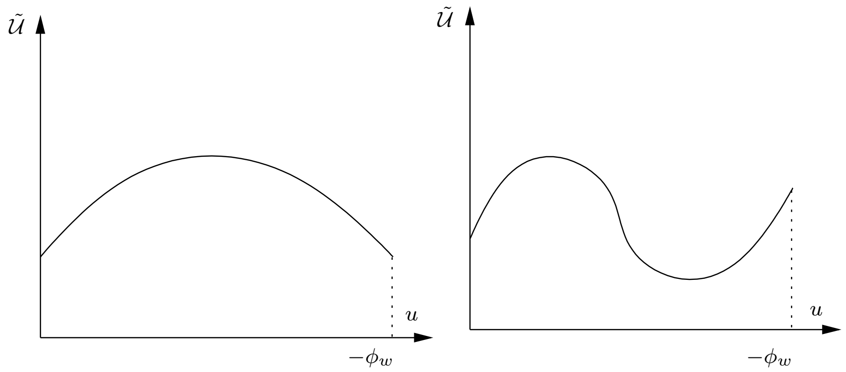
Proof. We begin with showing that $\tilde{\mathcal{U}}$ has at most two local minima. First of all, it is clear that $\tilde{\mathcal{U}}$ admits a local minimum (which is in fact a global one) since it is continuous over the compact set $[0, -\phi_w]$. We shall now distinguish two cases. Suppose $\mathcal{U}$ attains a local minimum at $u_1 \in (0, -\phi_w)$ then from corollary 2.6.21 it is unique in the interval $(0, -\phi_w)$ and from proposition 2.6.20 $\tilde{\mathcal{U}}$ is non decreasing over $[u_1, -\phi_w]$ and necessarily any other local minimum is attained at $u_2 = 0$. On the contrary if $\tilde{\mathcal{U}}$ does not have any local minimum in $(0, -\phi_w)$ this implies that $\tilde{\mathcal{U}}$ has at most two local minima attained at $u_1 = 0$ and $u_2 = -\phi_w$. By similar arguments, we can also show that $\tilde{\mathcal{U}}$ has at most two local maxima. □

It results from the above results that we can describe the variation of $\tilde{\mathcal{U}}$ depending on where local minima and maxima are located in $[0, -\phi_w]$. We give an illustration of all possible behavior for $\tilde{\mathcal{U}}$.



(a) $\tilde{\mathcal{U}}$ admits a unique local minimum and two local maxima.

(b) $\tilde{\mathcal{U}}$ admits a unique local minimum and unique local maximum.



(c) $\tilde{\mathcal{U}}$ has a unique local maximum and two local minima.

(d) $\tilde{\mathcal{U}}$ has one local maxima and one local minima in $(0, -\phi_w)$. It also has one local maximum at $u = -\phi_w$ and one local minimum at $u = 0$.

Figure 2.4 – Plots of all possible behavior of the function $u \in [0, -\phi_w] \mapsto \tilde{\mathcal{U}}(u)$. The function studied in the scope of theorem 2.6.3 is represented by the dashed line of Figure (B).

2.7 Numerical approximation of (NLP-M)

All the methods described here can be considered as standard tools. Our goal here is not to justify the approximations but only to give an idea on how the numerical methods have been implemented. It is a consequence of our construction of the solution, that the numerical approximation of the solution for the (VPA) system when the the potential ϕ is non increasing requires two independent step. Namely, given $\rho_0 \in \mathbb{R}$, $\alpha \in [0, 1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$:

1. we need to approximate the floating potential ϕ_w solution to (2.38) and construct an approximation of n_0 given by (2.37).
2. given $\varepsilon > 0$, we need to approximate the solution ϕ_ε to (NLP-M) by minimizing the function J_ε.

Note that in the scope of theorem 2.6.3, we were only able to prove the well-posedness of (NLP-M) in the case $\rho_0 = 0$ with f_i^{in} satisfying the kinetic Bohm criterion (2.51). Nevertheless, for the physical interpretation it is interesting to treat the case where the kinetic Bohm criterion does not hold. Theoretically, minimizers always exist and their approximation is standard. To make sure that they are solving (NLP-M), we shall make an a posteriori verification. For the first step, we shall use a standard Newton method to approximate the solution to the non linear equation (2.38). For the second one, we shall use a projected gradient algorithm [27].

2.7.1 Description of the numerical methods

Numerical quadrature for velocity integrals

Both step of the numerical approximation of the solution requires to compute velocity integrals. As a result, we need a rather good numerical approximation to compute the velocity integrals. For a target function $g : \mathbb{R}^+ \to \mathbb{R}$ that is $L^1(\mathbb{R}^+)$ and piecewise smooth we implement the following decomposition:

$$\int_{\mathbb{R}^+} g(v)dv = \int_{[v_{min},v_{max}]} g(v)dv + \int_{\mathbb{R}^+\setminus[v_{min},v_{max}]} g(v)dv,$$

where $[v_{min}, v_{max}]$ is the largest interval in $\mathbb{R}^+$ where $|g|$ is greater than some a priori bound chosen to be small. Then each integrals is approximated with quadrature formulas. Typically, for the first one we proceed as follows: for a piecewise smooth g, there is some $N \in \mathbb{N}$ and $v_{min} \leqslant v_1 < v_2 < ... < v_N \leqslant v_{max}$ such that g is smooth on each (v_k, v_{k+1}) for all $k \in \{1, .., N\}$ so that we can write

$$\int_{[v_{min},v_{max}]} g(v)dv = \int_{v_{min}}^{v_1} g(v)dv + \sum_{k=1}^{N-1} \int_{v_k}^{v_{k+1}} g(v)dv + \int_{v_N}^{v_{max}} g(v)dv.$$

Then for all $k \in \{1, .., N\}$, the integral on the segment $[v_k, v_{k+1}]$ is approximated via a Gauss-Legendre quadrature. Namely, given a quadrature $(w_i, \xi_i)_{i=1...,N_g} \subset [0, +\infty) \times (-1, 1)$ of order $2N_g - 1$ with $N_g \in \mathbb{N}^*$, we use the following approximation

$$\int_{v_k}^{v_{k+1}} g(v)dv \approx \frac{v_{k+1} - v_k}{2} \sum_{i=1}^{N_g} w_i g\left(\frac{(v_{k+1} - v_k)}{2}\xi_i + \frac{v_k + v_{k+1}}{2}\right).$$

For the second integral, we write

$$\int_{\mathbb{R}^+\setminus[v_{min},v_{max}]} g(v)dv = \int_0^{v_{min}} g(v)dv + \int_{v_{max}}^{+\infty} g(v)dv.$$

The integration on the segment $[0, v_{min}]$ is treated as previously with a Gauss-Legendre quadrature. The integration on $[v_{max}, +\infty)$ is approximated with a Gauss-Laguerre quadrature, namely given a quadrature $(w_i, \xi_i)_{i=1\ldots,N_g} \subset [0, +\infty) \times [0, +\infty)$ of order $2N_g - 1$ with $N_g \in \mathbb{N}^*$, we use the following approximation

$$\int_{v_{max}}^{+\infty} g(v)dv = \int_0^{+\infty} g(w + v_{max})dw \approx \sum_{i=1}^{N_g} w_i g(\xi_i + v_{max})e^{\xi_i}.$$

For a quantitative error analysis of the approximation, we refer the reader to [47].

Numerical approximation of the floating potential

The numerical approximation of the floating potential ϕ_w is done through a standard Newton algorithm. Indeed, given $\rho_0 \in \mathbb{R}$, $\alpha \in [0, 1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$, we consider the problem of finding the unique root of the function defined for all $\psi \leqslant 0$ by $\tilde{\mathcal{W}}(\psi) = \mathcal{W}(\psi) - \frac{\sqrt{2\pi}}{1-\alpha} \int_{\mathbb{R}^+} f_i^{in}(v)vdv$ where $\mathcal{W}$ is the function defined in (2.39). Since $\tilde{\mathcal{W}}' > 0$ on $(-\infty, 0]$ the Newton procedure is well-defined and we can define the sequence $(\phi_w^n)_{n\in\mathbb{N}} \subset (-\infty, 0]$ by induction

$$\begin{cases} \phi_w^0 \leqslant 0, \\ \phi_w^{n+1} = \phi_w^n - \frac{\tilde{\mathcal{W}}(\phi_w^n)}{\tilde{\mathcal{W}}'(\phi_w^n)}, \forall n \in \mathbb{N}. \end{cases}$$

Moreover, since $\mathcal{W}$ is $C^2(-\infty, 0)$, for ϕ_w^0 well chosen (in practice we start with $\phi_w^0 = 0$) this sequence converges to ϕ_w as $n \to +\infty$ [47]. The approximation of ϕ_w consists replacing $\tilde{\mathcal{W}}$ by its numerical approximation with velocity quadrature formulas denoted $I(\tilde{\mathcal{W}})$ and applying the previous Newton algorithm to it. The algorithm is stopped when the last computed term of the sequence is considered as a good approximation of the root of $I(\tilde{\mathcal{W}})$, that is for $\delta > 0$ small enough, the Newton procedure is stopped when there is $N_\delta \in \mathbb{N}$ such that $|I(\tilde{\mathcal{W}})(\phi_w^{N_\delta})| \leqslant \delta$, the number $\phi_w^{N_\delta}$ is then considered as an acceptable approximation of ϕ_w. The reference density n_0 defined by (2.37) is then approximated by replacing the exact floating potential by $\phi_w^{N_\delta}$, the velocity integrals are approximated by quadrature formulas as described above. The approximation of n_0 is denote $\tilde{n}_0$.

The projected gradient method for (NLP-M)

Let $\rho_0 \in \mathbb{R}$, $\alpha \in [0, 1)$ and $f_i^{in} \in \mathcal{I}_{ad}(\rho_0, \alpha)$ and $\varepsilon > 0$. We assume an approximation of the floating potential solution of (2.38) still denoted ϕ_w for the sake on conciseness. To compute numerically the solution to (NLP-M), we lift the boundary conditions and define $\bar{\phi} = \phi - x\phi_w$ where ϕ is solution to (NLP-M) where the floating potential is replaced by its approximation. We therefore consider the equivalent to (NLP-M) Poisson problem

$$\begin{cases} -\dfrac{\varepsilon^2}{2}\dfrac{d}{dx^2}\bar{\phi}(x) = -\dfrac{d}{d\psi}\bar{\mathcal{U}}(x, \bar{\phi}(x)) \quad \forall x \in (0, 1) \\ \bar{\phi}(0) = 0 \text{ and } \bar{\phi}(1) = 0 \\ \text{with } \bar{\mathcal{U}}(x, \psi) := \mathcal{U}(\psi + x\phi_w) \text{ for all } x \in [0, 1] \text{ and } \psi \in [(1 - x)\phi_w, -x\phi_w]. \end{cases}$$

Then we look for an approximation of minimizers of the functional defined for all $\bar{\phi} \in W = W(\rho_0, \alpha) := \left\{\bar{\phi} \in H_0^1(0, 1) \mid (1 - x)\phi_w \leqslant \bar{\phi} \leqslant -x\phi_w \text{ in } [0, 1]\right\}$ by

$$\bar{J}(\bar{\phi}) = \int_0^1 \left(\frac{\varepsilon^2}{2}|\bar{\phi}'(x)|^2 + \bar{\mathcal{U}}(x, \bar{\phi}(x))\right) dx.$$

Let $N \in \mathbb{N}^*$, the discretization consists of a mesh $(x_i := \frac{i}{(N+1)})_{i=0,\dots,N+1}$ with $h = \frac{1}{N+1}$ and the approximation of the Hilbert space $H_0^1(0,1)$ by a standard and conformous $\mathbb{P}_1$ finite element space V_0^h. More precisely

$$V_0^h := \left\{ \bar{\phi}_h \in C^0[0,1], \quad \bar{\phi}_h(0) = \bar{\phi}_h(1) = 0 \,|\, \forall i = 0,\dots,N \quad \bar{\phi}_h \underset{|[x_i,x_{i+1}]}{\in} \mathbb{P}_1, \right\} \tag{2.75}$$

and the admissible potential set is approximated by

$$W^h := \left\{ \bar{\phi}_h \in V_0^h \,|\, (1-x)\phi_w \leqslant \bar{\phi}_h \leqslant -x\phi_w \text{ in } [0,1] \right\}. \tag{2.76}$$

The discrete minimization problem then writes

$$\begin{cases} \text{Find } \bar{\phi}_h \in W^h \text{ such that} \\ \bar{\phi}_h = \arg \inf\limits_{\psi_h \in W^h} \bar{J}(\psi_h). \end{cases}$$

It is also well-posed and to approximate its minimizers, we use the projected gradient method. Namely, given $\eta > 0$ and $\delta > 0$, we compute iteratively

$$\begin{cases} \bar{\phi}_h^0 \in W^h \\ \bar{\phi}_h^{n+1} = \Pi\left(\bar{\phi}_h^n - \eta \nabla \bar{J}(\bar{\phi}_h^n)\right) \end{cases}$$

where $\Pi : V_0^h \to W^h$ is the projection on W^h for the $H_0^1(0,1)$ norm and $\nabla \bar{J}(\bar{\phi}_h^n) \in V_0^h$ is the gradient of $\bar{J}$ at $\bar{\phi}_h^n$, that is the unique solution of the variational problem

$$\begin{cases} \text{Find } u_h \in V_0^h \text{ such that for all } \psi_h \in V_0^h \\ (u_h, \psi_h)_{V_0} = d\bar{J}(\bar{\phi}_h^n)(\psi_h) \\ d\bar{J}(\bar{\phi}_h^n)(\psi_h) = \displaystyle\int_0^1 \left(\varepsilon^2 \frac{d}{dx}\bar{\phi}_h^n(x) \frac{d}{dx}\psi_h(x) + \frac{d}{d\psi}\bar{\mathcal{U}}(x, \bar{\phi}_h^n(x))\psi_h(x) \right) dx. \end{cases}$$

The algorithm is known to converge under C^1 regularity and convexity assumption on $\bar{J}$. These conditions are fulfilled when the boundary condition f_i^{in} satisfies the inequality (2.69). The reader eager to know more about the quantitative error analysis and the proof of convergence of the gradient projected method is refered to [27]. In practice, the algorithm is stopped when $\|\nabla \bar{J}(\bar{\phi}_h^n)\|_{H_0^1(0,1)} < \delta$ because in the scope of theorem 2.6.3, minimizers are also critical points.

Reconstruction of the distributions

Once the solution to (NLP-M) is approximated by a function ϕ_h, the distribution functions f_i and f_e are approximated respectively by the functions f_i^h and f_e^h defined respectively by

$$f_i^h(x,v) = \begin{cases} f_i^{in}(\sqrt{v^2 + 2\phi_h(x)}) \text{ if } (x,v) \in \{(x,v) \in [0,1]\times\mathbb{R} \,|\, v \geqslant \sqrt{-2\phi_h(x)}\}. \\ 0 \text{ elsewhere.} \end{cases}$$

$$f_e^h(x,v) = \begin{cases} \tilde{n}_0 f_e^{in}\left(\sqrt{v^2 - \frac{2}{\mu}\phi_h(x)}\right) \text{ if } (x,v) \in \{(x,v) \in [0,1]\times\mathbb{R} \,\mid\, v \geqslant -\sqrt{\frac{2}{\mu}\left(\phi_h(x) - \phi_w\right)}\} \\ \alpha\tilde{n}_0 f_e^{in}\left(\sqrt{v^2 - \frac{2}{\mu}\phi_h(x)}\right) \text{ elsewhere .} \end{cases}$$

where f_e^{in} is the semi-Maxwellian (2.31) and $\tilde{n}_0$ is an approximation of n_0.

2.7.2 Numerical results

We present two sets of numerical results. In the first one we perform numerical simulations that are in the scope of Theorem 2.6.3, that is in the case of a satisfied kinetic Bohm criterion. For these simulations we vary the parameters ε and α. In the second one, we perform numerical simulations with fixed values of ε and α but with an incoming ion boundary condition that violates the kinetic Bohm criterion (2.51). We consider the following physical parameters:

- We set the mass ratio $\mu = \frac{1}{3672}$ for a Deuterium plasma. It results in $\alpha_c \approx 0.95$.
- The boundary condition for the ions is $f_i^{in}(v) = \min(1, \dfrac{v^2}{\eta})\dfrac{e^{\dfrac{-(v-Z)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$ for all $v > 0$, where $\eta = 10^{-1}$ is a small parameter, $\sigma^2 = \frac{T_i}{T_e} = \frac{1}{2}$ is the temperature ratio and Z is a macroscopic velocity adjusted with respect to the kinetic Bohm criterion (2.51).
- We set the neutrality, that is $\rho_0 = 0$.
- We choose a mesh size $h = 2^{-11}$ and a tolerance parameter for our gradient algorithm $\delta = 10^{-6}$.

The case of a satisfied Bohm criterion

We fixed the value of Z chosen equal to $\frac{3}{2}$. The moments are computed numerically and we obtain:

$$\int_{\mathbb{R}+} f_i^{in}(v)dv \approx 0.99, \quad \int_{\mathbb{R}+} f_i^{in}(v)vdv \approx 1.4, \text{ and } \int_{\mathbb{R}+} \frac{f_i^{in}(v)}{v^2}dv \approx 0.74.$$

We can check numerically that both the admissibility condition (2.40) and the kinetic Bohm criterion (2.51) are satisfied. In figure 2.5 we have represented the ion incoming boundary condition f_i^{in}. We are

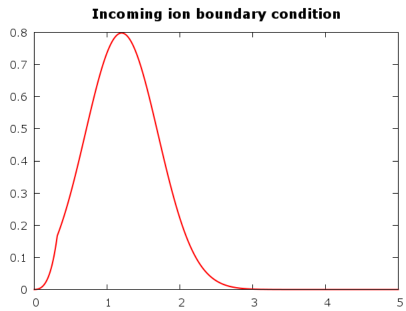


Figure 2.5 – Plot of the incoming ion boundary condition for positive velocities with $Z = \frac{3}{2}$.

now going to illustrate the behavior of the solution with respect to ε and α. We know from Theorem 2.6.3 that $n_i(x) \geqslant n_e(x)$ for all $0 \leqslant x \leqslant 1$. The general intuition is that when $\varepsilon > 0$ is small, one would expect n_i to be very close to n_e and ϕ_ε to be almost linear over some interval $[0, x^*(\varepsilon)]$ with $x^*(\varepsilon) > 0$. Then because of the potential drop the difference $n_i - n_e$ must become larger and larger as we approach the wall.

Case $\alpha = 0$ with varying ε We fix the re-emission coefficient $\alpha = 0$, the results are presented in figures 2.6,2.7,2.8, 2.9 and 2.10. The a priori bound (2.41) on ϕ_w gives $\phi_w \geqslant -2.80$ and the numerically computed wall potential is $\phi_w \approx -2.78$. The electron reference density is $n_0 \approx 0.50$. If figure 2.6 we have represented the graph of $\mathcal{U}$ over its definition domain $[\phi_w, 0]$. In agreement with the theory it is a decreasing function. For the data we have chosen it also seems to be convex.

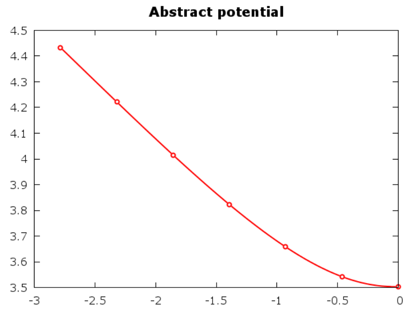


Figure 2.6 – Plot of the potential function $\mathcal{U}$ over $[\phi_w, 0]$ for $\alpha = 0$.

Figures 2.7,2.8, 2.9 and 2.10 represent the computed solution : f_i^ε, f_e^ε, n_i^ε, n_e^ε, ϕ_ε and $u_i^\varepsilon = \dfrac{\gamma_i^\varepsilon}{n_i^\varepsilon}$ for $\varepsilon \in \{1, 0.1, 0.01\}$. We observe that when ε is small a sheath of length of the order of ε develops near the wall and the sheath-edge denoted x^* varies with ε. In the simulations the sheath-edge corresponds to the point where approximately $|\phi_\varepsilon(x^*)| > 10^{-4}|\phi_w|$. We have $x^*(1) \approx 0$, $x^*(0.1) \approx 0.5$, $x^*(0.01) \approx 0.95$. For $x > x^*(\varepsilon)$, the plasma is significantly positively charged and there is a non negligible electric field that accelerates ions and decelerates electrons. Sufficiently fast electrons reach the wall and are absorbed. For $x \leqslant x_\varepsilon$, the plasma is almost neutral and there is no appreciable electric field, particles have constant velocities. These results are in good agreement with the physics, and moreover it confirms the commonly made assumption of semi-Maxwellian electron distribution function at the wall, see for example [44].

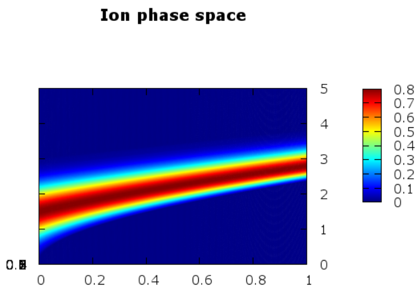
Case $\varepsilon = 0.1$ and varying α We fix $\varepsilon = 0.1$ and vary $\alpha \in \{0, 0.5, 0.9\}$. The results are qualitatively the same with the difference that some electrons are re-emitted from the wall with negative velocities, therefore we decide only to plot the electron and ion densities n_e^α and n_i^α, see figure 2.11. We also gather the different values of the wall potential ϕ_w and the electron reference density n_0 with respect to α in the table 2.1. The ion and electron densities seem to be respectively increasing functions of α.

α	ϕ_w	n_0
0	-2.7	0.5
0.5	-2.1	0.5
0.9	-0.48	0.5

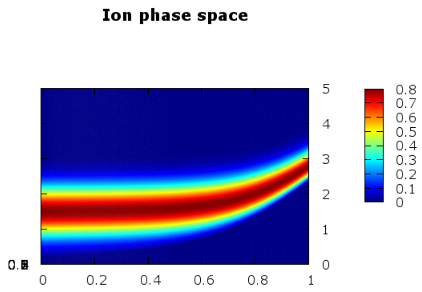
Table 2.1 – Values of the wall potential and the reference density for various values of α.

The case of a violated Bohm criterion

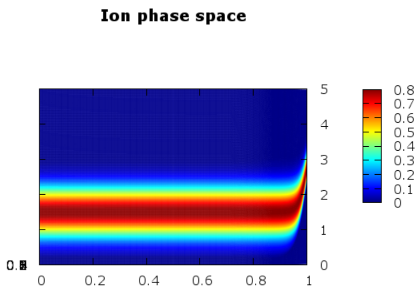
We present numerical results when the kinetic Bohm criterion (2.51) is not satisfied. We mention that we are not in the scope of Theorem 2.6.3, we cannot a priori ensure the existence and the uniqueness of a solution to (NLP-M). However, we can still minimize the functional J_ε and check a posteriori that the (numerically) computed minimizer is indeed a solution to (NLP-M). Consequently, for this numerical



(a) $\varepsilon = 1.0$



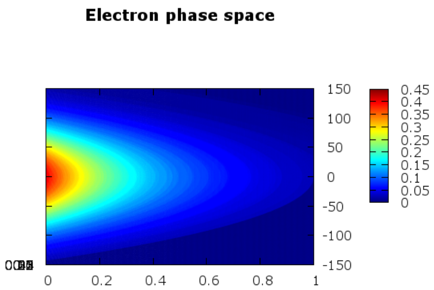
(b) $\varepsilon = 0.1$



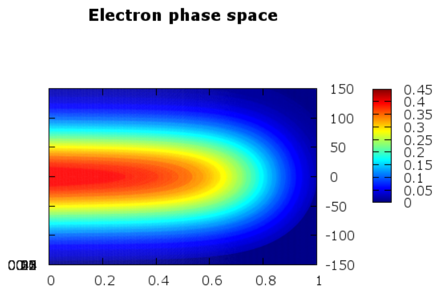
(c) $\varepsilon = 0.01$

Figure 2.7 – Ion distribution functions in the phase space for various ε. Plot (a),(b) and (c) are represented in the phase space $[0, 1] \times [0, 5]$.

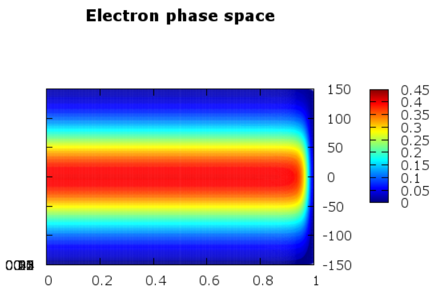
experiment we fix $\varepsilon = 0.01$, $\alpha = 0$. We choose $Z = 0.5$, for this value of Z the ion incoming boundary condition does not satisfy the Bohm criterion (2.51). Consequently, we know a priori that $\mathcal{U}''(0) < 0$ and thus the potential function $\mathcal{U}$ is locally concave near $\phi = 0$. In addition, because the slope at $\phi = 0$ is $\mathcal{U}'(0) = -\rho_0 = 0$ the potential function $\mathcal{U}$ is locally increasing near $\phi = 0$. Therefore the profile of $\mathcal{U}$ corresponds to one of the figure 2.4-(A). In figure 2.12 we have represented the function $\mathcal{U}$ over its domain of definition. The figure 2.13 represents respectively the ion and electron distribution function in the phase space. The figure 2.14 represents respectively the macroscopic densities and the electrostatic potential. We see that when the Bohm criterion is violated, there is two boundary layer, one is at $x = 0$ and the other one is at $x = 1$. The charge density is negative near $x = 0$ while it is positive near $x = 1$. We mention that since the point $x = 0$ is assumed to represent a position somewhere in the bulk plasma, it seems to us that the boundary layer at $x = 0$ is unphysical. Consequently, this unphysical effect shows a limitation of our model. Lastly, we mention that for other values of the macroscopic velocity Z which do not ensure the kinetic Bohm criterion (2.51) to be satisfied, the results are qualitatively the same.



(a) $\varepsilon = 1.0$



(b) $\varepsilon = 0.1$

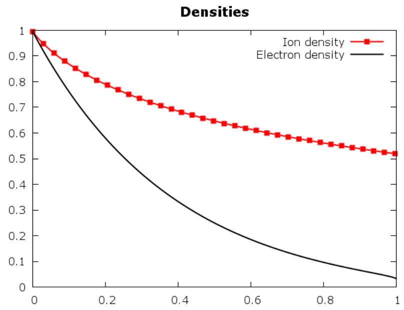


(c) $\varepsilon = 0.01$

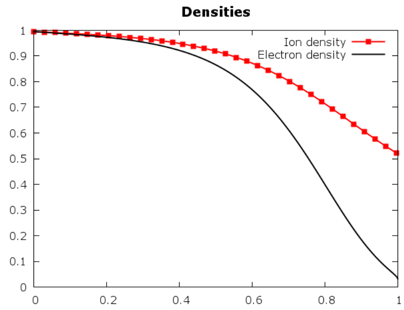
Figure 2.8 – Electron distribution functions in the phase space for various ε. Plot (a),(b) and (c) are represented in the phase space $[0, 1] \times [-150, 150]$.

2.8 Conclusion

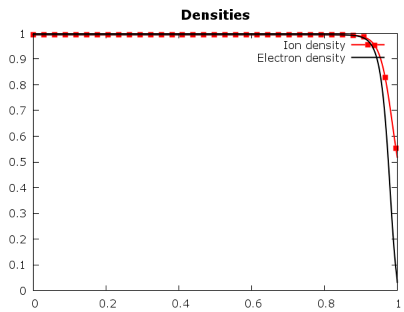
We have proposed and studied a stationary and one dimensional plasma-wall interaction model, based on a bi-kinetic description of ions and electrons. Due to the presence of the wall, the electron phase space density is represented by a truncated Maxwellian distribution. As for the ions, our model supports a large class of incoming velocity distributions f_i^{in} and we have shown that it is well posed under a moment condition on f_i^{in} which generalizes the usual Bohm criterion. Furthermore, we have identified a second condition that must be satisfied by f_i^{in} for the wall potential to be well-defined. Surprisingly enough, this second condition takes the form of an upper bound on the average velocity of the incoming ions but thanks to the large mass ratio μ we have verified that it is not in contradiction with the Bohm criterion. Our proof relies on a reformulation of the Vlasov-Poisson system into a non linear Poisson equation that we next study as a minimization problem. This approach also provides us with quantitative estimates for the boundary layer. A physically based sheath problem was then illustrated with numerical simulations. Results show that when the neutrality is assumed at $x = 0$ and when the incoming ion distribution is admissible and satisfies the kinetic Bohm criterion, then for a vanishing normalized Debye length ε a sheath of length of the order of ε develops at the wall. Out of the sheath the plasma is almost neutral while in the sheath it is not, ions are accelerated and electrons decelerated. These results provide a strong



(a) $\varepsilon = 1.0$

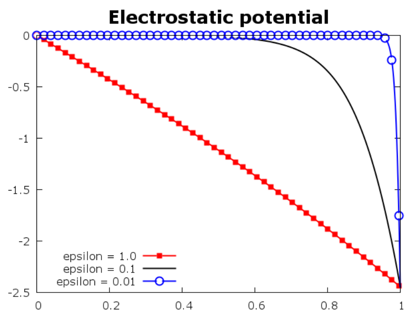


(b) $\varepsilon = 0.1$

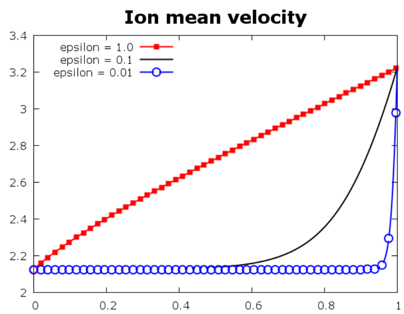


(c) $\varepsilon = 0.01$

Figure 2.9 – Ion and electron densities in space for various ε. Plot (a),(b),(c) are represented in the space $[0, 1]$.



(a) Electrostatic potential in space for various ε.



(b) Ion mean velocity in space for various ε.

Figure 2.10 – Electrostatic potential and ion mean velocity in the domain $[0, 1]$ for various values of ε.

numerical evidence for Theorem 2.6.3 and they are in good agreement with the simulations presented in [29]. We should add that this work takes full advantage of the one dimensional structure of our model. Although elementary, we hope this approach to be generic enough to be used in more general cases, including additional physics such as collision operators and magnetic fields [44, 39].

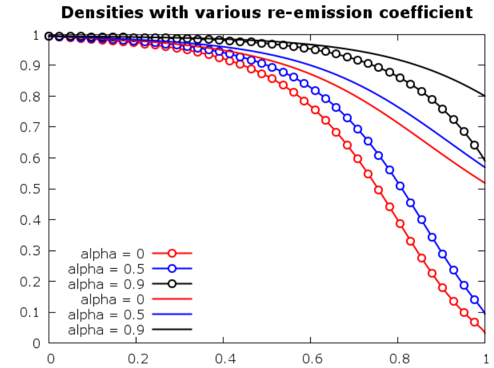


Figure 2.11 – Ion and electron densities for $\varepsilon = 0.1$ and $\alpha \in \{0, 0.5, 0.9\}$. Pointed lines correspond to electron densities and thick lines to ion densities.

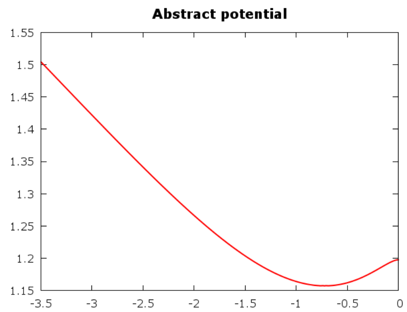
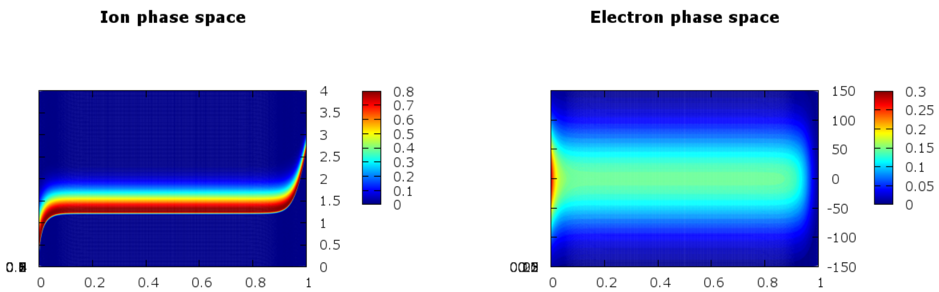


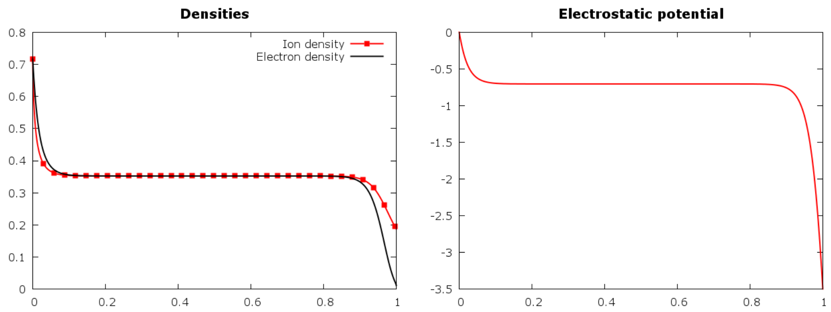
Figure 2.12 – Plot of the potential function $\mathcal{U}$ over $[\phi_w, 0]$ for $\alpha = 0$.



(a) Ion phase space

(b) Electron phase space

Figure 2.13 – Ion and electron distribution functions in the phase space. Plot (a) is represented in the phase space $[0, 1] \times [0, 5]$ while plot (b) is represented in the phase space $[0, 1] \times [-150, 150]$.



(a) Ion and electron densities in space. (b) Electrostatic potential in space.

Figure 2.14 – Macroscopic densities and electrostatic potential in space. Plot (a),(b) are represented in the space $[0, 1]$.

2.9 Appendix

Some fundamental results are reminded here for the self consistency of this work.

Theorem 2.9.1 (p. 135 [38]). *Let X be a reflexive Banach space, C a closed convex subset of X and $F : C \to \mathbb{R}$ a map. Moreover, assume*

1. *F is coercive, i.e $F(x) \to +\infty$ as $\|x\| \to +\infty$.*
2. *F is (sequentially) weakly lower semicontinuous, i.e for any sequences $(x_n)_{n\in\mathbb{N}} \subset C$ which converges to $x \in C$ for the weak topology, one has $x_n \rightharpoonup x \Rightarrow F(x) \leqslant \liminf F(x_n)$.*

then there exists $u \in C$ such that $F(u) := \inf\limits_{v\in C} F(v)$.

Lemma 2.9.2. *Let X and Y be two Banach spaces. Suppose X reflexive and $F : X \to Y$ is a compact mapping, then F is (sequentially) weakly-lower semicontinuous.*

The theory of Nemytskii operators provides continuity and differentiability results for some functional operators, see [4].

Definition 2.9.3. *Let be I a nonempty interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function. The Nemytskii operator associated with f is the map which associates to any measurable function $u : (0, 1) \to I$ the function $v := T_f(u)$ defined by $v(x) = f(u(x))$ for all $x \in (0, 1)$.*

Theorem 2.9.4. *Let be I a nonempty interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a continuous function over I then the Nemytskii operator associated with f, $T_f : C^0([0, 1]; I) \to C^0[0, 1]$ is continuous from $C^0([0, 1], I)$ to $C^0[0, 1]$.*

Theorem 2.9.5. *Let be I an nonempty interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a C^1-function over I then the Nemytskii operator associated with f, $T_f : C^0([0, 1], I) \to C^0[0, 1]$ is a C^1 mapping from $C^0([0, 1], I)$ to $C^0[0, 1]$ and its Fréchet derivative is given by*

$$dT_f(u)v = f'(u)v \quad \forall (u, v) \in C^0([0, 1], I) \times C^0[0, 1].$$

Chapter 3

Linear electrons stability for a bi-kinetic sheath model

3.1 Introduction

3.1.1 A kinetic model of plasma-wall dynamics: the Vlasov-Poisson-Ampère system

We consider an electrostatic and collisionless plasma consisting of one species of ions and electrons. We use a kinetic approach to model this plasma. To this purpose, we set $\Omega = (0,1) \times \mathbb{R}$ and denote by $(x, v) \in \overline{\Omega} = [0,1] \times \mathbb{R}$ the phase space variable, where x is the particle position and v the particle velocity. This work is concerned with the linear stability of an equilibrium for the two species Vlasov-Poisson system in the presence of spatial boundaries

$$\begin{cases} \partial_t f_i + v\partial_x f_i + E\partial_v f_i = 0 & \text{in} \quad (0,+\infty) \times \Omega, \\ \partial_t f_e + v\partial_x f_e - \frac{1}{\mu} E\partial_v f_e = 0 & \text{in} \quad (0,+\infty) \times \Omega, \end{cases} \tag{3.1}$$

$$-\varepsilon^2 \partial_{xx}\phi = n_i - n_e \quad \text{in} \quad (0,+\infty) \times (0,1), \tag{3.2}$$

where $f_i : [0,+\infty) \times \overline{\Omega} \to \mathbb{R}^+$, $f_e : [0,+\infty) \times \overline{\Omega} \to \mathbb{R}^+$ are the ions and electrons distribution functions in the phase-space and $\phi : [0,+\infty) \times [0,1] \to \mathbb{R}$ is the electric potential. Here the physical parameters μ and ε stand respectively for the mass ratio between electrons and ions, and a normalized Debye length that will be (for simplicity) in the sequel taken equal to 1. We also denote

$$E = -\partial_x \phi, \quad n_i = \int_{\mathbb{R}} f_i dv, \quad n_e = \int_{\mathbb{R}} f_e dv,$$

the electric field, the ion density and the electron density. The boundary conditions are given for all $t \in (0,+\infty)$ by

$$\begin{cases} f_i(t,0,v>0) = f_i^{in}(v), \quad f_i(t,1,v<0) = 0, \\ f_e(t,0,v>0) = f_e^{in}(v), \quad f_e(t,1,v<0) = \alpha f_e(t,1,-v), \end{cases} \tag{3.3}$$

$$\phi(t,0) = 0, \quad E(t,1) = E^*(t, f_i, f_e). \tag{3.4}$$

where f_i^{in} and f_e^{in} denote two given incoming particles velocity distributions that are time independent. The scalar parameter α belongs to the interval $[0,1)$ and represents the rate of re-emitted electrons in the domain $(0,1)$. The scalar $E^*(t,f_i,f_e)$ depends on the unknown (f_i,f_e,ϕ) via the formula

$$E^*(t,f_i,f_e) = \left(E_w^0 - \int_0^t \int_{\mathbb{R}} \left(f_i(\tau,1,v) - f_e(\tau,1,v)\right) v dv d\tau\right). \tag{3.5}$$

Up to our knowledge, the theory of existence and uniqueness for such a initial boundary value problem (3.1)-(3.4) has not been treated in full details. In the one dimensional case, there is the result of Bostan [13] which establishes the existence and uniqueness of the mild solution to a Vlasov-Poisson system in which the boundary conditions do not depend on the solution itself. Still in the one dimensional case, the work of Guo [31] studies the dynamic of a plane diode. Also, the result of BenAbdallah [9] shows the existence of weak solutions for the Vlasov-Poisson system in dimension greater than or equal to one, but once again the boundaries are not coupled to the solution itself. The existence and uniqueness of weak solutions in the half-space with specular reflection condition is obtained in [33]. The case of partially absorbing boundary condition is treated in [30]. The existence of a stationary solution to the system (3.1)-(3.4) was proven in [7], the stationary solution corresponds to the Debye sheath (see [63] for further physical details). This work can be considered as a continuation of the work [7], and a first step in the study of the wellposedness of the non-linear system (3.1)-(3.4).

3.1.2 Physical interpretation of the model

The bi-kinetic model (3.1)-(3.4) models the dynamical transition between the core of a plasma and a wall (see for instance [45]). The region of plasma we consider is modeled by the line segment $[0,1]$ where $x=0$ is assumed to be somewhere in the bulk plasma and thus a source of particles. The sources here are modeled by the injection of particles that are mathematically encoded in the given distributions f_i^{in} and f_e^{in}. The wall at $x=1$ is supposed to be metallic and partially absorbing: it absorbs completely the ions and re-emits a fraction α of the electrons. The parameter α can be seen as a constitutive parameter of the wall. The accumulation of charged particles at the wall induces an electric-field that is given by (3.5) (the number E_w^0 denotes the initial electric field at the wall). The boundary condition of the electric-field at the wall can be formally re-written as $\partial_t E(t,1)+j(t,1)=0$ where $j(t,1) := \int_{\mathbb{R}}(f_i(t,1,v)-f_e(t,1,v))vdv$ is the current density at the wall. At a formal level, one easily verifies that this boundary condition ensures the compatibility of the solutions to (3.1)-(3.4) with the Vlasov-Ampère system made of equations (3.1), (3.3) with the Maxwell-Ampère equation

$$\varepsilon^2 \partial_t E = -j \text{ in } (0,+\infty)\times[0,1], \quad j := \int_{\mathbb{R}} (f_i - f_e) v dv. \tag{3.6}$$

provided the initial data satisfy the Poisson equation

$$\partial_x E(0,.) = \int_{\mathbb{R}} f_i(0,.,.) - f_e(0,.,.) dv.$$

Because of this equivalence, we shall rather consider the Vlasov-Ampère system (3.1), (3.3) and (3.6).

3.1.3 Statement of the main result

The mathematical and physical aspect we investigate in this work is the linear stability of the Debye sheath for the Vlasov-Ampère model (3.1),(3.3) and (3.6). The rigorous mathematical construction of such an equilibrium was obtained for the first time for the model (3.1), (3.3) together with (3.6) in [7]. The main result of this work can be roughly summarized as follows: if the initial data that is a small perturbation of the sheath equilibrium, then the solution of the system (3.1),(3.3) and (3.6) remains close to the sheath equilibrium for all times provided the ions are frozen. To make things more precise at this stage, we need supplementary materials. Let us denote by $(f_i^\infty, f_e^\infty, \phi^\infty)$ the sheath equilibrium to (3.1),(3.3) and (3.6). Let us write the solution of (3.1),(3.3) and (3.6) as the sum of the sheath equilibrium plus an interior perturbation that affects only the electrons and the electrostatic field, namely: $(f_i, f_e, \phi) = (f_i^\infty, f_e^\infty + \tilde{f}_e, \phi^\infty + \tilde{\phi})$. The formal linearization yields the linearized Vlasov-Ampère system (after dropping the $\tilde{}$)

$$\text{(LVA)}: \begin{cases} \partial_t f_e + D f_e = E \partial_v f_e^\infty, & \text{in } (0,+\infty) \times \Omega \\ \partial_t E = \displaystyle\int_{\mathbb{R}} f_e v dv, & \text{in } (0,+\infty) \times [0,1] \\ f_e(t,0,v>0) = 0, \quad f_e(t,1,v<0) = \alpha f_e(t,1,-v) & \text{in } (0,+\infty) \end{cases}$$

where D denotes the first order linear differential operator defined formally by

$$D := v\partial_x - E^\infty \partial_v \text{ with } E^\infty = -(\phi^\infty)' \text{ being the equilibrium electric field.} \tag{3.7}$$

Because the equilibrium density f_e^∞ is discontinuous across the curve of equation $v = v_e(x)$ where the function v_e is defined for all $x \in [0,1]$ by

$$v_e(x) := -\sqrt{2\,(\phi^\infty(x) - \phi_w)}, \tag{3.8}$$

its velocity derivative takes the form

$$\partial_v f_e^\infty = [f_e^\infty]\delta^{v_e} - v f_e^\infty, \tag{3.9}$$

where δ^{v_e} is a Dirac distribution supported on the curve of equation $v = v_e(x)$, namely:

$$\langle \delta^{v_e}, \varphi \rangle = \int_0^1 \varphi(x, v_e(x)) dx \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{3.10}$$

and where

$$[f_e^\infty] := \lim_{v \to v_e(x)^+} f_e^\infty(x,v) - \lim_{v \to v_e(x)^-} f_e^\infty(x,v) \tag{3.11}$$

denotes the constant jump of f_e^∞ across the characteristic curve $v = v_e(x)$. As a consequence, it is natural to look for solutions to (LVA) that decompose into a singular part plus a regular one, as

$$f_e = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v) \tag{3.12}$$

with η_e and g_e two functions. The main result can be in rough terms stated as follows: *For any initial data (f_e^0, E^0) with f_e^0 of the form $f_e^0(x,v) = \eta^0(x)\delta^{v_e} + g_e^0(x,v)$ where η_0 and g_e^0 are two functions, there exists a couple of functions (η_e, g_e) and an electric field E such that if we define $f_e(t,x,v) = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v)$, then the couple (f_e, E) is solution to the linearized Vlasov-Ampère system with initial condition $f_e(t=0,x,v) = f_e^0(x,v)$ and $E(t=0,x) = E^0(x)$. Moreover the non negative energy functional defined for $t \geqslant 0$ by*

$$\mathcal{E}(t) = \frac{1}{2}\left(\int_0^1 \frac{\eta_e^2(t,x)|v_e(x)|dx}{[f_e^\infty]} dx + \int_\Omega \frac{g_e(t,x,v)^2}{f_e^\infty(x,v)} dx dv + \int_0^1 E(t,x)^2 dx\right) \tag{3.13}$$

is non increasing.

3.1.4 The mathematical approach and its difficulty

The main difficulty in the analysis lies in the fact that for $\alpha \in [0,1)$ the electron sheath equilibrium is a discontinuous function of the particle energy. This is due to the absorption of particles with positive velocities at the wall ($x = 1$), which creates a discontinuity that propagates into the domain along the characteristic curve $v = v_e(x)$. Thus the linearization of the Vlasov-Ampère system (3.1),(3.3),(3.6) around the sheath equilibrium $(f_i^\infty, f_e^\infty, E^\infty)$ with no perturbation on the ions, yields a linear system whose solution still denoted (f_e, E) is singular. Assuming a decomposition of the form

$$f_e(t,x,v) = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v)$$

where η_e and g_e are two functions and δ^{v_e} is a Dirac mass supported by the characteristic curve of equation $v = v_e(x)$ yields another linear system on (η_e, g_e, E). Making a suitable change of variable leads to the system (VAL). The system (VAL) contains a degenerate transport equation because the given velocity field v_e vanishes at $x = 1$ and its derivate $\partial_x v_e$ is not essentially bounded as the Diperna-Lions theory of transport equation [23] requires. This difficulty is overcomed using the fact that the velocity field is only weakly degenerated, it vanishes at the border like a squareroot. This allows us to prove a Hardy-Poincaré type inequality. Ultimately by applying the Hille-Yosida theorem, we show that the linearized system (VAL) is well-posed and that the energy of the system is non increasing.

3.1.5 Previous works

Stability issues for Vlasov-Poisson systems in bounded geometry are of a tremendous importance for practical applications, be it in the modeling of laboratory plasmas, or in the design of numerical methods. Unfortunately, and despite its worthy interest, it seems that it has not been studied in full details. Stability analysis for such a Vlasov-Poisson system has already been performed in the absence of spatial boundaries, that is, either in all space or in a periodical setting [56, 50, 41]. However, it seems that in the presence of spatial boundaries, the question of stability has not been extensively adressed. Up to our knowledge, it is only very recently, with the work of Nguyen and Strauss in [51] that the question was raised. The authors considered the Vlasov-Maxwell system in a cylindrical geometry and assumed an equilibrium that is a C^1 function of the particle energy and momentum. Additionally, they assumed as in the previous works [56, 41] that the equilibrium, on its support, is monotone in the particle energy, which seem to be a key ingredient to prove stability results. Stability issues related to non monotone equilibria not seem to be completely understood, we mention the seminal work of Morrison [49] where he gives sufficient conditions for stability, and more rencently the work of Ben-Artzi [10]. In [10], the author gives sufficient conditions for an instability, but on the other hand the analysis is performed in an unbounded geometry.

3.1.6 Organization of the chapter

The plan of this chapter is as follows. In section 3.2, we derive the system (VAL) and give a precise statement of the main result, including the functional spaces, and the notions of solutions we consider. In section 3.3, we state the Hardy-Poincaré type inequality and give some technical lemmas needed to prove the main result. We eventually prove the main result. In the appendix 3.4, we briefly discuss the regularity of the solution.

3.2 The stability result

3.2.1 Description of the sheath equilibrium

The equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$ is associated with an electron boundary condition which is a Maxwellian, namely it takes the form

$$f_e^{in}(v) = n_0\sqrt{\frac{2}{\pi}}e^{-\frac{v^2}{2}} \text{ for } v > 0. \tag{3.14}$$

where $n_0 > 0$ is an electron reference density. The sheath equilibrium is a stationary and weak solution of the Vlasov-Ampère system (3.1), (3.3), (3.6). It belongs to the space $(L^1 \cap L^\infty(\Omega))^2 \times C^2[0,1]$ and enjoys the following properties:

1. For all $x \in [0,1]$, $(\phi^\infty)''(x) \leqslant 0$, $E^\infty(x) := -(\phi^\infty)'(x) \geqslant 0$, $\phi^\infty(0) = 0$, $\phi^\infty(1) =: \phi_w$ and $E^\infty(1) =: E_w^\infty > 0$.
2. $f_i^\infty(x,v) = \begin{cases} f_i^{in}\left(\sqrt{v^2 + 2\phi^\infty(x)}\right) & \text{for } (x,v) \text{ s.t } v > \sqrt{-2\phi^\infty(x)} \\ 0 \text{ elsewhere.} \end{cases}$
3. $f_e^\infty(x,v) = n_0\sqrt{\frac{2}{\pi}}\begin{cases} e^{-\frac{v^2}{2}}e^{\phi^\infty(x)} & \text{for } (x,v) \text{ s.t } v \geqslant v_e(x) \\ \alpha e^{-\frac{v^2}{2}}e^{\phi^\infty(x)} & \text{for } (x,v) \text{ s.t } v < v_e(x), \end{cases}$
4. $\int_{\mathbb{R}} f_i^\infty(x,v)v dv = \int_{\mathbb{R}} f_e^\infty(x,v)v dv$ for all $x \in [0,1]$.

Such an equilibrium is proven to exist in chapter 2, under the necessary and sufficient condition that the following kinetic Bohm criterion

$$\frac{\displaystyle\int_{\mathbb{R}+} \frac{f_i^{in}(v)}{v^2} dv}{\displaystyle\int_{\mathbb{R}+} f_i^{in}(v) dv} < \frac{\left(\sqrt{2\pi} + (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} \frac{e^{-\frac{v^2}{2}}}{v^2} dv\right)}{\left(\sqrt{2\pi} - (1-\alpha)\displaystyle\int_{\sqrt{-2\phi_w}}^{+\infty} e^{-\frac{v^2}{2}} dv\right)}.$$

holds true for $f_i^{in} \in L^1 \cap L^\infty(\mathbb{R}^+)$. The physical meaning of this inequality is that there cannot be too many ions particles entering the domain with low velocities. It is instructive to have a representation of the ions and electrons characteristics in the phase space (see Figure 3.1). We see that for the electrons, the equilibrium is a truncated Maxwellian distribution. Especially, it is discontinuous accross the characteristic curve $S := \{(x, v_e(x)) \,/\, x \in [0,1]\}$ where the function v_e is defined in (3.8). To be more precise we have the following:

Lemma 3.2.1. *a)* $f_e^\infty \in C^2(\Omega \setminus S)$.

b)

$$\partial_v f_e^\infty = [f_e^\infty]\delta^{v_e} - v f_e^\infty.$$

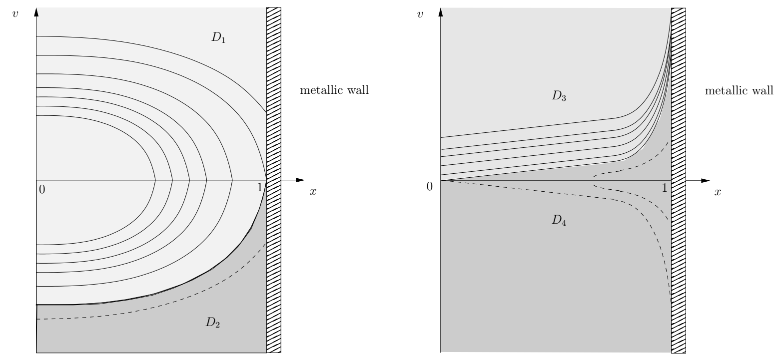
where $[f_e^\infty] = n_0\sqrt{\frac{2}{\pi}}(1-\alpha)e^{\phi_w} > 0$ *is the jump* f_e^∞ *across the characteristic curve* S *defined in* (3.11) *and* δ^{v_e} *is the Dirac mass supported by the function* v_e *defined in* (3.8).

Proof. a) It is straightforward from its definition that f_e^∞ belongs to $C^2(\Omega \setminus S)$. b) It follows from an

Figure 3.1 – Ions and electrons phase space

(a) Schematic characteristic electron trajectories associated with the potential ϕ^∞. The electron equilibrium f_e^∞ is discontinuous across the curve $v = v_e(x)$ (bold curve).

(b) Schematic characteristic ions trajectories associated with the potential ϕ^∞. The ion equilibrium f_i^∞ vanishes outside D_3 (lighter gray).



integration by parts. Indeed, for all $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}
\langle \partial_v f_e^\infty, \varphi \rangle =& - \langle f_e^\infty, \partial_v \varphi \rangle \\
=& - \int_0^1 \int_{\mathbb{R}} f_e^\infty(x,v) \partial_v \varphi(x,v) dv dx \\
=& - \int_0^1 \int_{v \geqslant v_e(x)} f_e^\infty(x,v) \partial_v \varphi(x,v) dv dx \\
& - \int_0^1 \int_{v < v_e(x)} f_e^\infty(x,v) \partial_v \varphi(x,v) dv dx \\
=& - n_0 \sqrt{\frac{2}{\pi}} \int_0^1 \left[e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} \varphi(x,v) \right]_{v_e(x)}^{+\infty} dx \\
& - n_0 \sqrt{\frac{2}{\pi}} \int_0^1 \int_{v \geqslant v_e(x)} v e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} \varphi(x,v) dv dx \\
& - n_0 \sqrt{\frac{2}{\pi}} \int_0^1 \left[\alpha e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} \varphi(x,v) \right]_{-\infty}^{v_e(x)} dx \\
& - n_0 \sqrt{\frac{2}{\pi}} \int_0^1 \int_{v < v_e(x)} \alpha v e^{-\frac{v^2}{2}} e^{\phi^\infty(x)} \varphi(x,v) dv dx.
\end{aligned}$$

It is then easy to conclude. □

Lemma 3.2.2. *The function $x \in [0,1] \mapsto v_e(x) := -\sqrt{2(\phi^\infty(x) - \phi_w)}$ has the following properties:*

a) $v_e \in C^0([0,1]) \cap C^2([0,1))$

b) For all $x \in [0,1)$, $v_e(x) < 0$, $v_e(1) = 0$, $v_e(x) \underset{x \to 1^-}{\sim} -\nu\sqrt{1-x}$ where $\nu = \sqrt{2E_w^\infty}$.

c) For all $x \in (0,1)$, $v_e'(x)v_e(x) + E^\infty(x) = 0$.

d) $v_e \in W^{1,1}(0,1)$ and $\frac{1}{v_e} \in L^1(0,1)$.

Proof. We skip the proof because it is essentially a consequence of the regularity of ϕ^∞. □

This lemma is important because it makes precise the regularity of v_e which must be known insofar as it will appear as the velocity field of a one dimensional transport equation of the form $(\partial_t + v_e\partial_x)u = s$. The fact that $\frac{1}{v_e} \in L^1(0,1)$ implies that the characteristics, namely the solutions of the ode

$$\dot{x}(t) = v_e(x(t))$$

have a finite incoming time into the domain $[0,1]$. We give a sketch of the characteristics in the plan (x,t).

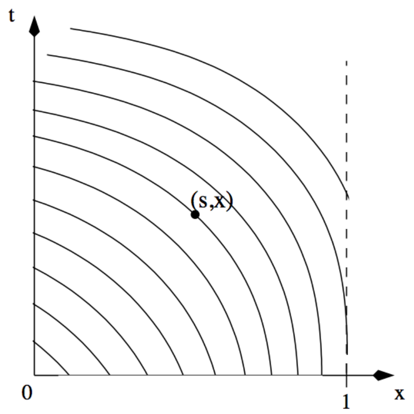


Figure 3.2 – Characteristics curves in the plan (x,t). The backward in time characteristics cross $x = 1$ in finite time.

3.2.2 Derivation of the linearized system (VAL)

We now derive the linear system (VAL). Let us give the definition of solution we consider for the linearized Vlasov-Ampère system.

Definition 3.2.3. *Assume f_e^0 a measure on $\overline{\Omega}$ and $E^0 \in L^2(0,1)$. Let f_e be a measure on $[0,+\infty) \times \overline{\Omega}$ and $E \in W^{1,\infty}_{loc}\left([0,+\infty); L^2(0,1)\right)$. We say that (f_e, E) is a weak solution of (LVA) system iff:*

a) For all $\varphi \in C_c^1([0,+\infty) \times \overline{\Omega})$ such that $\varphi(t,0,v \leqslant 0) = 0$ and $\varphi(t,1,v \geqslant 0) = -\alpha\varphi(t,1,-v)$

$$\begin{aligned}
&- \langle f_e, \partial_t\varphi + D\varphi\rangle_{[0,+\infty)\times\overline{\Omega}} \\
&= \langle f_e^0, \varphi(0,.)\rangle_{\overline{\Omega}} \\
&+ [f_e^\infty] \int_0^{+\infty}\int_0^1 E(t,x)\varphi(t,x,v_e(x))dxdt \\
&- \int_0^{+\infty}\int_0^1\int_{\mathbb{R}} E(t,x)vf_e^\infty(x,v)\varphi(t,x,v)dxdvdt.
\end{aligned}$$

b) The current density $(t,x) \in (0,+\infty) \times (0,1) \mapsto j_e(t,x) := \langle f_e(t,x,.), v\rangle_{\mathbb{R}}$ *belongs to*

$$L^\infty_{loc}([0,+\infty); L^2(0,1)),$$

$$\partial_t E(t,x) = j_e(t,x) \text{ for a.e } (t,x) \in (0,+\infty) \times (0,1) \text{ and } E(t=0,x) = E^0(x) \text{ for a.e } x \in (0,1).$$

One readily checks that f_e satisfies

$$\partial_t f_e + Df_e = [f_e^\infty]E\delta^{v_e} - Evf_e^\infty \text{ in } \mathcal{D}'((0,+\infty)\times\Omega). \tag{3.15}$$

Since the right hand side of (3.15) is the sum of Dirac mass supported by v_e plus a function, a natural idea is to look for $f_e \in \mathcal{D}'((0,+\infty)\times\Omega)$ under the following form

$$f_e = \eta_e(t,x)\delta^{v_e} + g_e(t,x,v)$$

where $\eta_e : [0,+\infty)\times[0,1] \to \mathbb{R}$ and $g_e : [0,+\infty)\times\overline{\Omega}$ are a priori two regular functions. Note that for all $\varphi \in \mathcal{D}((0+\infty)\times\Omega)$

$$\langle f_e, \varphi\rangle = \int_0^{+\infty}\int_0^1 \eta_e(t,x)\varphi(t,x,v_e(x))dxdt + \int_0^{+\infty}\int_\Omega g_e(t,x,v)\varphi(t,x,v)dxdvdt$$

Proposition 3.2.4. *Let* (f_e, E) *be a weak solution of (LVA) with* f_e *of the form* (3.12) *where* $\eta_e \in L^1_{loc}([0,+\infty)\times(0,1))$ *and* $g_e \in L^1_{loc}([0,+\infty)\times\Omega)$*. Then* f_e *is solution of the Vlasov equation* (3.15) *if and only if* η_e *and* g_e *satisfy*

$$\partial_t\eta_e + \partial_x(v_e\eta_e) = [f_e^\infty]E \text{ in } \mathcal{D}'((0,+\infty)\times(0,1)), \tag{3.16}$$

$$\partial_t g_e + Dg_e = -Evf_e^\infty \text{ in } \mathcal{D}'((0,+\infty)\times\Omega). \tag{3.17}$$

Proof. We omit the proof because it follows from standard calculations. □

Let us now introduce the change of unknown

$$w_e(t,x) := v_e(x)\eta_e(t,x), \quad h_e(t,x,v) := \begin{cases} \frac{g_e(t,x,v)}{\sqrt{f_e^\infty(x,v)}} & \text{if } f_e^\infty(x,v) \neq 0 \\ g_e(t,x,v) & \text{if } f_e^\infty(x,v) = 0. \end{cases} \tag{3.18}$$

We recall that $Df_e^\infty = 0$ in $\mathcal{D}'(\Omega)$ and we note that by definition f_e^∞ never vanishes when $\alpha \in (0,1)$. One also checks by a straightforward calculation that the couple (η_e, g_e) is a solution of (3.16)-(3.17) if and only if (w_e, h_e) is a solution to

$$\partial_t w_e + v_e\partial_x w_e = [f_e^\infty]v_e E \text{ in } \mathcal{D}'((0,+\infty)\times(0,1)), \tag{3.19}$$

$$\partial_t h_e + Dh_e = -Ev\sqrt{f_e^\infty} \text{ in } \mathcal{D}'((0,+\infty)\times\Omega). \tag{3.20}$$

The transport equation (3.19) has a negative on $[0,1)$ and vanishing at $x=1$ velocity field v_e defined by (3.8). It is a priori not clear wether a boundary condition is needed for this equation. Having a closer look at the characteristics that are draw in figure ??, we see that a boundary condition at $x=1$ is necessary if one wants to solve the equation on the domain $(0,+\infty)\times(0,1)$. Physically speaking, the number $w_e(t,x)$ is the current of electrons at the position $x \in [0,1]$ carried by the singular part of f_e notably

$$w_e(t,x) = \eta_e(t,x)\langle\delta_{v=v_e(x)}, v\rangle$$

where $\delta_{v=v_e(x)}$ denotes the classical Dirac mass supported at $v = v_e(x)$. For $\eta_e \underset{x\to 1^-}{=} o(\frac{1}{v_e})$ this yields $w_e(t,1) = 0$. We thus impose the homogeneous Dirichlet boundary condition $w_e(t,1) = 0$. The boundary conditions for the transport equation (3.20) are easily derived from the original boundary condition on f_e, they write for all $t > 0$:

$$h_e(t,0,v>0) = 0, \quad h_e(t,1,v<0) = \sqrt{\alpha}h_e(t,1,-v). \tag{3.21}$$

The initial boundary value problem then writes: given $w_e^0 : [0,1] \to \mathbb{R}$, $h_e^0 : \overline{\Omega} \to \mathbb{R}$ and $E^0 \in [0,1] \to \mathbb{R}$, find $w_e : [0,+\infty) \times [0,1] \to \mathbb{R}$, $h_e : [0,+\infty) \times \overline{\Omega} \to \mathbb{R}$ and $E : [0,+\infty) \times [0,1] \to \mathbb{R}$ solution to

$$(\text{VAL}) : \begin{cases} \partial_t w_e + v_e \partial_x w_e = [f_e^\infty] v_e E \text{ in } (0,+\infty) \times (0,1), \\ \partial_t h_e + D h_e = -Ev\sqrt{f_e^\infty} \text{ in } (0,+\infty) \times \Omega, \\ \partial_t E = w_e + \int_{\mathbb{R}} v\sqrt{f_e^\infty} h_e dv \text{ in } (0,+\infty) \times [0,1], \\ h_e(t>0,0,v>0) = 0, \ h_e(t>0,1,v<0) = \sqrt{\alpha}h_e(t,1,-v), \\ w_e(t>0,1) = 0, \end{cases}$$

satisfying $w_e(t=0,.) = w_e^0$, $h_e(t=0,.,.) = h_e^0$, $E(t=0,.) = E^0$.

3.2.3 The main result

We are now in position to state precisely our main result. First let us begin with defining what linear stability stands for in this work.

Definition 3.2.5. *Let G and H be two Hilbert spaces equipped respectively with the norm $\|.\|_H$, $\|.\|_G$ and with the continuous embedding $G \hookrightarrow H$. We say that the equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$ is linearly stable by interior electron perturbation of the form (3.12) iff:*

a) For all $(w_e^0, h_e^0, E^0) \in G$ the system (VAL) admits a unique strong solution $(w_e, h_e, E) \in C^0([0,+\infty); G) \cap C^1([0,+\infty); H)$.

b) For all $\epsilon > 0$, there is $\eta > 0$ such that $\|(w_e^0, h_e^0, E^0)\|_H < \eta \Rightarrow \|(w_e, h_e, E)\|_H < \epsilon \quad \forall t \geqslant 0$.

The Hilbert spaces to be considered are the following

$$G := H^1_{v_e^{\frac{1}{2}},0}(0,1) \times W^2_{0,\alpha}(\Omega) \times L^2(0,1), \quad H := L^2_{v_e^{-\frac{1}{2}}}(0,1) \times L^2(\Omega) \times L^2(0,1)$$

where the spaces $L^2_{v_e^{-\frac{1}{2}}}(0,1)$, $H^1_{v_e^{\frac{1}{2}},0}(0,1)$ and $W^2_{0,\alpha}(\Omega)$ are defined in section 3.3.

Theorem 3.2.6. *The equilibrium $(f_i^\infty, f_e^\infty, \phi^\infty)$ is linearly stable in the sense of Definition 3.2.5. More precisely,*

a) For all $(w_e^0, h_e^0, E^0) \in G$ there exists a unique strong solution $(w_e, h_e, E) \in C^0([0,+\infty); G) \cap C^1([0,+\infty); H)$ to (VAL).

b) The energy defined in (3.13) is well-defined and for all times $t \geqslant 0$, re-writes

$$\mathcal{E}(t) = \frac{1}{2}\left(\int_0^1 \frac{w_e(t,x)^2}{[f_e^\infty]|v_e(x)|}dx + \int_\Omega h_e(t,x,v)^2 dxdv + \int_0^1 E(t,x)^2 dx\right)$$

and and is non increasing.

One of the key ingredient in the proof of the Theorem 3.2.6 is the following energy identity.

Proposition 3.2.7 (Energy dissipation)**.** *Strong solutions to (VAL) satisfy*

$$\frac{d}{dt}\mathcal{E}(t) = -\frac{1}{2}\left(\frac{1}{[f_e^\infty]}w_e^2(t,0) + (1-\alpha)\int_{\mathbb{R}^+} vh_e^2(t,1,v)dv - \int_{\mathbb{R}^-} vh_e^2(t,0,v)dv\right) \leqslant 0.$$

In particular, for all $0 \leqslant t \leqslant t'$, $0 \leqslant \mathcal{E}(t') \leqslant \mathcal{E}(t)$.

Proof. Let $(w_e, h_e, E) \in C^0([0,+\infty); G) \cap C^1([0,+\infty); H)$ a solution to (VAL). We set $\mathcal{E}_{w_e}(t) := \frac{1}{[f_e^\infty]}\int_0^1 \frac{w_e(t,x)^2}{|v_e(x)|}dx$, $\mathcal{E}_{h_e}(t) := \int_\Omega h_e(t,x,v)^2 dxdv$ and $\mathcal{E}_{pot}(t) := \int_0^1 E(t,x)^2 dx$. We compute each terms separately. Because of the regularity of (w_e, h_e, E) we can differentiate under the integral sign.

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{w_e}(t) =& \frac{2}{[f_e^\infty]}\int_0^1 \frac{\partial_t w_e(t,x) w_e(t,x)}{|v_e(x)|}dx \\ =& \frac{2}{[f_e^\infty]}\int_0^1 \left([f_e^\infty]E(t,x)v_e(x) - v_e(x)\partial_x w_e(t,x)\right)\frac{w_e(t,x)}{|v_e(x)|}dx\end{aligned}$$

Once again because $w_e(t,.) \in H^1_{v_e^{\frac{1}{2}},0}(0,1) \subset L^2_{v_e^{-\frac{1}{2}}}(0,1)$ the second integral is convergent

$$\left|\int_0^1 (v_e(x)\partial_x w_e(t,x))\frac{w_e(t,x)}{|v_e(x)|}dx\right| < +\infty,$$

and we deduce

$$\frac{d}{dt}\mathcal{E}_{w_e}(t) = -2\int_0^1 E(t,x)w_e(x)dx - \frac{1}{[f_e^\infty]}w_e^2(t,0).$$

We now compute the energy part associated with h_e. Using the boundary conditions and an integration by parts we get

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{h_e}(t) =& 2\int_\Omega \partial_t h_e(t,x,v)h_e(t,x,v)dxdv \\ =& 2\int_\Omega \left(-E(t,x)v\sqrt{f_e^\infty(x,v)} - Df_e^\infty(x,v)\right)h_e(t,x,v)dxdv \\ =& -2\int_\Omega E(t,x)v\sqrt{f_e^\infty(x,v)}h_e(t,x,v)dxdv \\ & -(1-\alpha)\int_{\mathbb{R}^+} vh_e^2(t,1,v)dv + \int_{\mathbb{R}^-} vh_e^2(t,0,v)dv.\end{aligned}$$

Note that since $h_e(t,.,.) \in W^2_{0,\alpha}(\Omega)$ the boundary terms make sense. We lastly turn to the electric part of the energy.

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{pot}(t) =& 2\int_0^1 \partial_t E(t,x)E(t,x)dx \\ =& 2\int_0^1 \left(w_e(t,x) + \int_{\mathbb{R}} v\sqrt{f_e^\infty(x,v)}h_e(t,x,v)dv\right)E(t,x)dx.\end{aligned}$$

Gathering all terms together enables to get the desired identity. In particular, we deduce that $t \in [0,+\infty) \mapsto \mathcal{E}(t)$ is non increasing. Hence $\mathcal{E}(t) \leqslant \mathcal{E}(t')$ for all $0 \leqslant t \leqslant t'$. $\square$

3.3 Functional spaces, technical lemmas and proof of the main result

In this section, we define the functional framework that is part of the main result 3.2.6. We eventually prove the main result by showing that the Hill-Yosida theorem applies.

3.3.1 Functional spaces

We define the following spaces

$$H^1_{v_e^{\frac{1}{2}}}(0,1) := \{u \in L^2(0,1) \text{ s.t } \sqrt{|v_e|}u' \in L^2(0,1)\}$$

where v_e is the function defined by (3.8) and note that it is such that $\frac{1}{v_e} \in L^1(0,1)$. It is a Hilbert space endowed with the inner product

$$(u,v)_{H^1_{v_e^{\frac{1}{2}}}(0,1)} := \int_0^1 u(x)v(x) + |v_e(x)|u'(x)v'(x)dx \quad \forall (u,v) \in H^1_{v_e^{\frac{1}{2}}}(0,1)^2.$$

Moreover, we can prove the imbedding $H^1_{v_e}(0,1) \hookrightarrow C^0[0,1]$ so that we can define the space

$$H^1_{v_e^{\frac{1}{2}},0}(0,1) := \{u \in H^1_{v_e^{\frac{1}{2}}}(0,1) \text{ s.t } u(1) = 0\}.$$

We also defined the weighted Lebesgue space

$$\mathcal{L}^2_{v_e^{-\frac{1}{2}}}(0,1) := \{u : (0,1) \to \mathbb{R} \text{ measurable s.t } \int_0^1 \frac{u^2(x)}{|v_e(x)|}dx < +\infty\}$$

and the quotient space

$$L^2_{v_e^{-\frac{1}{2}}}(0,1) := \mathcal{L}^2_{v_e^{-\frac{1}{2}}}(0,1)/\mathcal{R}$$

where $\mathcal{R}$ denotes the usual equivalence relation of almost everywhere equality for the Lebesgue measure. The space $L^2_{v_e^{-\frac{1}{2}}}(0,1)$ is an Hilbert space endowed with the inner product

$$(u,v)_{L^2_{v_e}(0,1)} := \int_0^1 \frac{u(x)}{\sqrt{|v_e(x)|}} \frac{v(x)}{\sqrt{|v_e(x)|}}dx \quad \forall (u,v) \in L^2_{v_e^{-\frac{1}{2}}}(0,1)^2.$$

An important tool in this work is the following inequality.

Lemma 3.3.1 (Hardy-Poincaré type inequality)**.** *For all* $\varphi \in H^1_{v_e^{\frac{1}{2}},0}(0,1)$,

$$\int_0^1 \frac{\varphi(x)^2}{|v_e(x)|}dx \leqslant \|\frac{1}{v_e}\|^2_{L^1(0,1)} \int_0^1 |v_e(x)|\varphi'(x)^2dx.$$

Consequently, $H^1_{v_e^{\frac{1}{2}},0}(0,1) \hookrightarrow L^2_{v_e^{-\frac{1}{2}}}(0,1)$.

Proof. We prove the inequality for $\varphi \in C_c^\infty(0,1)$. Let $\delta > 0$, one has for all $x \in [0, 1-\delta]$

$$\varphi(x) - \varphi(1-\delta) = -\int_x^{1-\delta} \frac{\varphi'(s)\sqrt{|v_e(s)|}}{\sqrt{|v_e(s)|}} ds.$$

Using the Cauchy-Schwarz inequality yields

$$|\varphi(x) - \varphi(1-\delta)| \leqslant \|\sqrt{|v_e|}\varphi'\|_{L^2(x,1-\delta)} \|\frac{1}{\sqrt{|v_e|}}\|_{L^2(x,1-\delta)}$$
$$\leqslant \left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx\right)^{\frac{1}{2}} \|\frac{1}{v_e}\|^{\frac{1}{2}}_{L^1(0,1)}.$$

Taking the limit as $\delta \to 0^+$ yields for all $x \in [0,1]$.

$$|\varphi(x)| \leqslant \left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx\right)^{\frac{1}{2}} \|\frac{1}{v_e}\|^{\frac{1}{2}}_{L^1(0,1)}.$$

Therefore for all $x \in [0,1)$ we have

$$\frac{\varphi(x)^2}{|v_e(x)|} \leqslant \frac{1}{|v_e(x)|}\left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx\right) \|\frac{1}{v_e}\|_{L^1(0,1)}.$$

One has therefore for $\delta > 0$

$$\int_0^{1-\delta} \frac{\varphi(x)^2}{|v_e(x)|} dx \leqslant \int_0^{1-\delta} \frac{1}{|v_e(x)|} dx \left(\int_0^1 |v_e(x)|\varphi'(x)^2 dx\right) \|\frac{1}{v_e}\|_{L^1(0,1)}$$
$$\leqslant \|\frac{1}{v_e}\|^2_{L^1(0,1)} \int_0^1 |v_e(x)|\varphi'(x)^2 dx.$$

Taking the limit at $\delta \to 0^+$ yields the desired inequality. The result extends to functions of the space $H^1_{v_e^{\frac{1}{2}},0}(0,1)$ by using the density of $C_c^\infty(0,1)$ in $H^1_{v_e^{\frac{1}{2}},0}(0,1)$ (see [16] Lemma 2.6 for the proof of density). $\square$

Thanks to Lemma 3.3.1 we can define on $H^1_{v_e^{\frac{1}{2}},0}(0,1)$ the following norm

$$\|u\|_{H^1_{v_e^{\frac{1}{2}},0}(0,1)} := \left(\int_0^1 |v_e(x)|u'(x)^2 dx\right)^{\frac{1}{2}} \quad \forall u \in H^1_{v_e^{\frac{1}{2}},0}(0,1).$$

It is an equivalent norm to the $H^1_{v_e^{\frac{1}{2}}}(0,1)$-norm. We will also need the following density result.

Lemma 3.3.2. *$H^1_{v_e^{\frac{1}{2}},0}(0,1)$ is dense in $L^2_{v_e^{-\frac{1}{2}}}(0,1)$.*

Proof. Let $u \in L^2_{v_e^{-\frac{1}{2}}}(0,1)$. Let us build a sequence $(u_n)_{n\in\mathbb{N}} \subset H^1_{v_e^{\frac{1}{2}},0}(0,1)$ such that $\|u_n - u\|_{L^2_{v_e^{-\frac{1}{2}}}(0,1)} \underset{n\to+\infty}{\to} 0$. We firstly remark that since $\frac{u}{\sqrt{-v_e}} \in L^2(0,1)$ and because $H^1_0(0,1)$ is dense in $L^2(0,1)$ there is a sequence $(\tilde{u}_n)_{n\in\mathbb{N}} \subset H^1_0(0,1)$ such that $\int_0^1 \left|\tilde{u}_n - \frac{u}{\sqrt{-v_e}}\right|^2(x)dx \underset{n\to+\infty}{\to} 0$. Let us then define for all $n \in \mathbb{N}$

$u_n := \tilde{u}_n\sqrt{-v_e}$. To conclude the proof, it suffices to check that for all $n \in \mathbb{N}$, $u_n \in H^1_{v_e^{\frac{1}{2}},0}(0,1)$. Because $v_e \in C^0[0,1]$ one readily verifies that $u_n \in C^0[0,1] \cap L^2(0,1)$. Let us now compute the derivative. One has in $\mathcal{D}'(0,1)$

$$u_n' = \tilde{u}_n'\sqrt{-v_e} - \tilde{u}_n\frac{v_e'}{2\sqrt{-v_e}} \Rightarrow \sqrt{-v_e}u_n' = -\tilde{u}_n v_e - \frac{\tilde{u}_n v_e'}{2}.$$

One has $\tilde{u}_n v_e \in L^2(0,1)$, it therefore suffices to prove that $\tilde{u}_n v_e' \in L^2(0,1)$. Using Lemma 3.2.2 d) we have for all $x \in (0,1)$

$$\tilde{u}_n(x)v_e'(x) = -\tilde{u}_n(x)\frac{E^\infty(x)}{v_e(x)}.$$

Since $E^\infty \in C^0[0,1]$, it suffices to show that $\dfrac{\tilde{u}_n}{v_e} \in L^2(0,1)$. Still using Lemma 3.2.2 b), we have $\dfrac{\tilde{u}_n^2(x)}{v_e^2(x)} \underset{x\to 1^-}{\sim} \nu^2\dfrac{\tilde{u}_n(x)^2}{(1-x)}$. But $\tilde{u}_n \in H_0^1(0,1)$ and a classical Hardy inequality (see [14, p.147] for instance) enables us to conclude that $\dfrac{\tilde{u}_n}{v_e} \in L^2(0,1)$. $\square$

We now define

$$W^2(\Omega) := \left\{h \in L^2(\Omega) \text{ s.t } Dh \in L^2(\Omega)\right\}.$$

Following [8], for a function $h \in W^2(\Omega)$ we can define its restriction to $\Sigma_- := \{0\}\times(0,+\infty)\cup\{1\}\times(-\infty,0)$ and $\Sigma_+ := \{0\}\times(-\infty,0)\cup\{1\}\times(0,+\infty)$. Moreover, $h_{|\Sigma_-}$ and $h_{|\Sigma_+}$ belongs respectively to $L^2_{loc}(\Sigma_-)$ and $L^2_{loc}(\Sigma_+)$. Thus, we can also define

$$W^2_{0,\alpha}(\Omega) := \left\{h \in W^2(\Omega) \text{ s.t } h(0,v>0) = 0 \text{ and } h(1,v<0) = \sqrt{\alpha}h(1,-v)\right\}.$$

Lemma 3.3.3. *$W^2_{0,\alpha}(\Omega)$ is dense in $L^2(\Omega)$ for the L^2-norm.*

Proof. It suffices to remark that $C^1_c(\Omega) \subset W^2_{0,\alpha}(\Omega)$. Then we deduce that $\overline{C^1_c(\Omega)} \subset \overline{W^2_{0,\alpha}(\Omega)} \subset L^2(\Omega)$ where $\overline{X}$ denotes the closure of the set X in $L^2(\Omega)$ for the L^2-norm. But, $C^1_c(\Omega)$ is dense in $L^2(\Omega)$ so $\overline{C^1_c(\Omega)} = L^2(\Omega)$ and then $\overline{W^2_{0,\alpha}(\Omega)} = L^2(\Omega)$. $\square$

To finish with this section, we state a Lemma due to Bardos [8] and give a Green formula.

Lemma 3.3.4. *$C^\infty_c(\overline{\Omega}) \cap W^2_{0,\alpha}(\Omega)$ is dense in $W^2_{0,\alpha}(\Omega)$ for the norm defined by*

$$\|h\|^2_{W^2_{0,\alpha}} := \|h\|^2_{L^2(\Omega)} + \|Dh\|^2_{L^2(\Omega)} \quad \forall h \in W^2_{0,\alpha}.$$

Lemma 3.3.5. *(Traces integrability) Let $h \in W^2_{0,\alpha}(\Omega)$ then $h(1,.) \in L^2(\mathbb{R}^+,|v|dv)$ and $h(0,.) \in L^2(\mathbb{R}^-,|v|dv)$ and the following Green Formula holds :*

$$\int_\Omega Dh(x,v)h(x,v)dxdv = (1-\alpha)\int_{\mathbb{R}^+}\frac{v}{2}h^2(1,v)dv - \int_{\mathbb{R}^-}\frac{v}{2}h^2(0,v)dv.$$

Proof. We argue by density. Let $h \in W^2_{0,\alpha}(\Omega)$ then in virtue of Lemma 3.3.4 there is $(h_n)_{n\in\mathbb{N}} \subset C^\infty_c(\overline{\Omega}) \cap W^2_{0,\alpha}(\Omega)$ such that $\|h_n - h\|_{W^2_{0,\alpha}} \underset{n\to+\infty}{\to} 0$. Using the boundary conditions and a Green Formula (that is valid for regular functions) we have for all $n \in \mathbb{N}$,

$$\int_\Omega Dh_n(x,v)h_n(x,v)dxdv = (1-\alpha)\int_{\mathbb{R}^+}\frac{v}{2}h_n^2(1,v)dv - \int_{\mathbb{R}^-}\frac{v}{2}h_n^2(0,v)dv.$$

By standard arguments, it is easy to see that

$$\int_0^1 \int_{\mathbb{R}} Dh_n(x,v)h_n(x,v)dxdv \underset{n\to+\infty}{\to} \int_0^1 \int_{\mathbb{R}} Dh(x,v)h(x,v)dxdv.$$

Then the sequences $\left(\int_{\mathbb{R}^+} \frac{v}{2} h_n^2(1,v)dv\right)_{n\in\mathbb{N}}$ and $\left(\int_{\mathbb{R}^-} \frac{v}{2} h_n^2(0,v)dv\right)_{n\in\mathbb{N}}$ are bounded and converge (up to an extraction). Lastly, we can show that the trace operators

$$\gamma_0 : h \in C_c^\infty(\overline{\Omega}) \cap W^2_{0,\alpha}(\Omega) \mapsto h(0,.) \in L^2(\mathbb{R}^-, |v|dv),$$

$$\gamma_1 : h \in C_c^\infty(\overline{\Omega}) \cap W^2_{0,\alpha}(\Omega) \mapsto h(1,.) \in L^2(\mathbb{R}^+, |v|dv)$$

extend both continuously to $W^2_{0,\alpha}(\Omega)$. This finally enables us to pass to the limit at both side of the previous equality so that the formula holds. □

We lastly mention that by a similar density argument we can also prove the following Green-formula.

Lemma 3.3.6. *For all $h \in W^2_{0,\alpha}(\Omega)$ and for all $\psi \in W^2(\Omega)$ such that $\psi(0, v<0) = 0$ and $\psi(1, v>0) = \sqrt{\alpha}\psi(1,-v)$ a.e we have,*

$$\int_\Omega Dh(x,v)\psi(x,v)dxdv = -\int_\Omega h(x,v)D\psi(x,v)dxdv.$$

3.3.2 Proof of the main result

We prove the main result by checking that the Hille-Yosida's Theorem applies (see [14, p.105] or the appendix 3.6 for a precise statement). The Hilbert space $H = L^2_{v_e^{-\frac{1}{2}}}(0,1) \times L^2(\Omega) \times L^2(0,1)$ is equipped with the inner product

$$(U_1,U_2)_H := \frac{1}{[f_e^\infty]}(w_1,w_2)_{L^2_{v_e^{-\frac{1}{2}}}(0,1)} + (h_1,h_2)_{L^2(\Omega)} + (E_1,E_2)_{L^2(0,1)},$$

for all $U_1 := (w_1,h_1,E_1)$, $U_2 := (w_2,h_2,E_2)$ in H. We introduce the unbounded operator $A : D(A) \subset H \to H$ defined by :

$$\begin{cases} AU := \begin{pmatrix} v_e\partial_x w - [f_e^\infty]v_e E \\ Dh + Ev\sqrt{f_e^\infty} \\ -\left(w + \int_{\mathbb{R}} v\sqrt{f_e^\infty} h dv\right) \end{pmatrix}, \\ \forall U := \begin{pmatrix} w \\ h \\ E \end{pmatrix} \in D(A) = H^1_{v_e^{\frac{1}{2}},0}(0,1) \times W^2_{0,\alpha}(\Omega) \times L^2(0,1), \end{cases}$$

Lemma 3.3.7. *The unbounded operator $A : D(A) \subset H \to H$ has the following properties:*

a) It is dissipative, in the sense that $(AU,U)_H \geqslant 0$.

b) $D(A)$ is dense in H.

Proof. a) We prove that A is dissipative. Let $U := \begin{pmatrix} w \\ h \\ E \end{pmatrix} \in D(A)$. We compute

$$
\begin{aligned}
(AU, U)_H = \, & \underbrace{\frac{1}{[f_e^\infty]} \left(v_e \partial_x w - \frac{[f_e^\infty]}{\mu} v_e E, w \right)_{L^2_{v_e^{-\frac{1}{2}}}(0,1)}}_{:=I_1} \\
& + \underbrace{\left(Dh + Ev\sqrt{f_e^\infty}, h \right)_{L^2(\Omega)}}_{:=I_2} \\
& - \underbrace{\left(w + \int_{\mathbb{R}} hv\sqrt{f_e^\infty} dv, E \right)_{L^2(0,1)}}_{:=I_3}.
\end{aligned}
$$

We can compute I_1 and I_2 by an integration by parts so that we obtain :

$$I_1 = \int_0^1 \left(-\partial_x w(x) + [f_e^\infty] E(x) \right) w(x) dx = \frac{w^2(0)}{2} + [f_e^\infty] \int_0^1 E(x) w(x) dx.$$

$$
\begin{aligned}
I_2 &= \int_0^1 \int_{\mathbb{R}} \left(Dh(x,v) + E(x) v \sqrt{f_e^\infty(x,v)} \right) h(x,v) dx dv \\
&= (1-\alpha) \int_{\mathbb{R}^+} \frac{vh^2(1,v)}{2} dv - \int_{\mathbb{R}^-} \frac{vh^2(0,v)}{2} dv + \int_\Omega E(x) v \sqrt{f_e^\infty(x,v)} h(x,v) dx dv.
\end{aligned}
$$

$$I_3 = \int_0^1 E(x) w(x) dx + \int_\Omega E(x) v \sqrt{f_e^\infty(x,v)} h(x,v) dx dv.$$

Collecting all terms together, we finally deduce

$$(AU, U)_H = \frac{1}{[f_e^\infty]} \frac{w^2(0)}{2} + (1-\alpha) \int_{\mathbb{R}^+} \frac{vh^2(1,v)}{2} dv - \int_{\mathbb{R}^-} \frac{vh^2(0,v)}{2} dv \geqslant 0.$$

b) The fact that $D(A)$ is dense in H is a consequence of Lemmas 3.3.2 and 3.3.3. □

Lemma 3.3.8. *The unbounded operator $I + A : D(A) \subset H \to H$ is such that $R(I + A) = H$.*

Proof. We are going to apply the Proposition 3.6.2. Of course, $D(A)$ is dense in H as we have already proven. Since H is a Hilbert space, by the Riesz representation theorem, we can identify H with its dual, namely $H' \cong H$ so that the adjoint operator of $I + A$ is the unbounded operator $(I + A)^* : D(A^*) \subset$

$H \to D(A)'$ defined by:

$$\begin{cases} (I+A)^*U := U + A^*U \text{ with } A^*U := \begin{pmatrix} -v_e\partial_x w + [f_e^\infty]v_e E \\ -Dh - Ev\sqrt{f_e^\infty} \\ \left(w + \int_{\mathbb{R}} v\sqrt{f_e^\infty} h dv\right) \end{pmatrix}, \\ \forall U := \begin{pmatrix} w \\ h \\ E \end{pmatrix} \in D(A^*) = \{u \in H^1_{v_e^{\frac{1}{2}}}(0,1) \text{ s.t } u(0) = 0\} \times \\ \{h \in W^2(\Omega) \text{ s.t } h(0, v<0) = 0 \text{ and } h(1, v>0) = \sqrt{\alpha} h(1,-v)\} \times L^2(0,1) \end{cases}$$

Of course, $D(A^*)$ is also dense in H. This enables to prove that $I + A$ is closed (see [14, Proposition II.16, p.28]). Lastly, straightforward integrations by parts allow us to prove that A^* is also dissipative which in the end turns out to be sufficient to prove that

$$((I+A)^*U, U)_H = \|U\|_H^2 + (A^*U, U)_H \geqslant \|U\|_H^2.$$

Therefore a Cauchy-Schwarz inequality yields $\|(I+A)^*U\|_H \geqslant \|U\|_H$. Therefore the Proposition 3.6.2 applies, it concludes the proof. □

The proof of the main result is now straightforward. Combining Lemmas 3.3.7 and 3.3.8 we can apply the Hille-Yosida theorem. For all $(w_e^0, h_e^0, E^0) \in D(A)$ there is a unique $\begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} \in C^0([0,+\infty); D(A)) \cap C^1([0,+\infty), H)$ such that

$$\frac{d}{dt}\begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} + A\begin{pmatrix} w_e \\ h_e \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with

$$\begin{pmatrix} w(0,.) \\ h(0,.,.) \\ E(0,.) \end{pmatrix} = \begin{pmatrix} w_e^0 \\ h_e^0 \\ E^0 \end{pmatrix}.$$

The boundary conditions are included in the space $D(A)$ so that they are satisfied by the solution. The solution also satisfies for all times $0 \leqslant t \leqslant t'$ the inequality

$$\left\| \begin{pmatrix} w_e(t',.,.) \\ h_e(t',.,.) \\ E(t',.) \end{pmatrix} \right\|_H^2 \leqslant \left\| \begin{pmatrix} w_e(t,.,.) \\ h_e(t,.,.) \\ E(t,.,.) \end{pmatrix} \right\|_H^2$$

which re-writes in terms of the energy $\mathcal{E}(t') \leqslant \mathcal{E}(t)$. It also implies the stability with respect to perturbation. Indeed, for any $\epsilon > 0$, it suffices to choose (w_e^0, h_e^0, E^0) such that $\mathcal{E}(0) < \epsilon$ to get that $\mathcal{E}(t) < \epsilon$ for all times $t \geqslant 0$.

3.4 Comments on the regularity of η_e

We want to explain why we have worked on the flux variable w_e rather than on the density number η_e. First notice that one has the following imbedding

$$H^1_{v_e^{\frac{1}{2}},0}(0,1) \hookrightarrow C^0[0,1]$$

which implies that $w_e(t,.) \in C^0[0,1]$ for all $t \geqslant 0$. The boundary condition on w_e therefore makes sense. As far as the electron density η_e is concerned, we now observe that because $\frac{1}{v_e} \in L^1(0,1)$ one has

$$\eta_e \in C^0\left([0,+\infty); C^0[0,1) \cap L^1(0,1)\right) \cap C^1\left([0,+\infty); L^1(0,1)\right),$$

and $\eta_e \underset{x\to 1^-}{=} o(\frac{1}{v_e})$. However, it is not clear wether η_e can be extended by continuity at $x = 1$. In fact, we cannot expect any more integrability on the spatial derivative of η_e and thus the notion of boundary condition at $x = 1$ is not obvious. To illustrate this lack of regularity, a simple calculation shows that

$$v_e \partial_x \eta_e = \partial_x w_e - \frac{v_e'}{v_e} w_e \text{ in } \mathcal{D}'(0,1).$$

The first term at the right hand side of the equality belongs to $L^1(0,1)$ while the second term does not :

$$\forall t \geqslant 0\, \forall x \in (0,1) \quad |\frac{v_e'(x)}{v_e(x)} w_e(t,x)| \geqslant \min_{x\in[0,1]} |w_e(t,x)||\frac{v_e'(x)}{v_e(x)}|,$$

and $\int_0^1 |\frac{v_e'(x)}{v_e(x)}|dx = +\infty$. This lack of integrability of $\partial_x \eta_e$ is inherent in the nature of the functional space $H^1_{v_e^{\frac{1}{2}},0}(0,1)$. For instance, we can show that a function of the form $w : x \in [0,1] \mapsto (1-x)^s$ belongs to $H^1_{v_e^{\frac{1}{2}},0}(0,1)$ if and only if $s > \frac{1}{4}$. The quotient $\eta := \frac{w}{v_e}$ has the following behavior $\eta(x) \underset{x\to 1^-}{\sim} \frac{-(1-x)^{s-\frac{1}{2}}}{\nu}$ with $s - \frac{1}{2} > -\frac{1}{4}$. Then for $\frac{1}{4} < s < \frac{1}{2}$ we have $\lim_{x\to 1^-} \eta(x) = -\infty$ and $\eta \in L^1(0,1)$ but $\partial_x \eta \notin L^1(0,1)$.

3.5 Conclusion

We have proposed a one dimensional bi-kinetic model of the dynamical transition between a plasma and a wall. In our model ions and electrons transport are modeled by Vlasov equations with a self-consistent electric-field that satisfies the Maxwell-Ampère equation. We have studied the stability of an equilibrium for this system, that is the so-called Debye sheath. Especially, we have shown that when ions are fixed the electron equilibrium is linearly stable. The proof relies on the linearization the solution of the Vlasov-Ampère system near the electron equilibrium together with the decay of an energy functional for the linearized system. We paid a special attention to the mathematical structure of the solution for linearized system that turns out to be singular. Especially, we focus on providing an adequate functional framework to justify the energy identity and prove the wellposedness of the linearized system by checking that the Hille-Yosida theorem applies. Although elementary, we consider this work as preliminary step in the study of the stability of both ions and electrons. We nevertheless mention that our stability result is in connection with the mathematical structure of our electron equilibrium. The electron equilibrium in our case is piecewise defined, and on each piece it is as a decreasing function of the microscopic energy. This seems to be a key ingredient for the stability, and it was already used in previous works [56, 41].

Because of the Bohm criterion, it is not the case for the ions and thus the question of stability seems more intricate.

3.6 Appendix

Definition 3.6.1. *Let H a Hilbert space and $A : D(A) \subset H \to H$ an unbounded linear operator. We say that A is dissipative if*

$$(Av, v)_H \geqslant 0 \quad \forall v \in D(A),$$

A is maximal dissipative if moreover $R(I + A) = H$.

We recall a result that characterizes surjective operators.

Proposition 3.6.2 ([14] Theorem II.19 page 30)**.** *Let $A : D(A) \subset H \to H$ an unbounded linear operator, closed with $D(A)$ dense in H. Then*

$$R(A) = H \Leftrightarrow \exists C \geqslant 0 \text{ such that } \|v\|_{H'} \leqslant C\|A^* v\|_{H'} \quad \forall v \in D(A^*),$$

where A^* is the adjoint-operator of A and H' is the dual space of H.

Theorem 3.6.3 (Hille-Yosida)**.** *Let A be a maximal dissipative operator in a Hilbert space H. Then for all $u_0 \in D(A)$ there is a unique $u \in C^1([0, +\infty); H) \cap C^0([0, +\infty); D(A))$ solution of the problem :*

$$\begin{cases} \frac{du}{dt} + Au = 0 \text{ on } [0, +\infty) \\ u(0) = u_0. \end{cases}$$

Moreover, one has

$$\|u(t)\|_H \leqslant \|u_0\| \text{ and } \|\frac{du}{dt}(t)\|_H \leqslant \|Au_0\|_H \text{ for all } t \geqslant 0.$$

Chapter 4

A bi-kinetic model of plasma sheath: the magnetized case

4.1 Introduction

When a constant external magnetic field making a non zero angle θ of incidence with the wall is considered, an additional region that precedes the Debye sheath settles. This region, known as the Chodura sheath [19] is a quasi neutral region of several ion Larmor radii in thickness. While the Debye sheath is essentially electrostatic in the sense that the dominant part of the force is the electric field, in the Chodura sheath both the electric and the external magnetic field are in competition. As compared with the purely electrostatic case, in the seminal work of Chodura it was shown numerically that the formation of these two regions requires that the ion mean velocity in the orthogonal direction to the wall satisfies at the entrance of the magnetic presheath, the Bohm-Chodura inequality

$$u_i^x \geqslant c_s \cos(\theta) \tag{4.1}$$

where $\theta \in [0, \frac{\pi}{2})$ denotes the angle of incidence of the magnetic field. In his work, Chodura considers a model for both ions and electrons and the numerical results show that the electrostatic potential could have spatial oscillations but decreases as we approach the wall. In particular, the graph of the electric potential is contained in a decreasing enveloped curve. It is also pointed out that the magnitude of the drop over the magnetic presheath depends on the angle θ, namely if we denote $\Delta\Phi_{mp}$ the difference of potential between the entrance of the magnetic presheath and the entrance the of the Debye sheath, it was shown to satisfy the empirical estimate

$$\left|\frac{q\Delta\Phi_{mp}}{kT_e}\right| \propto |\ln(\cos(\theta))|. \tag{4.2}$$

These results were later confirmed numerically in [17, 44] using kinetic models.

However efficient, the inequality (4.2) is empirical and has no theoretical foundation, especially it degenerates in the case of a tangent to the wall magnetic field $\theta = \frac{\pi}{2}$ which seems to show off a limitation of the model considered by Chodura. It also raises the natural question of knowing whether the analysis made in[19, 44, 62] can be extended to that case. Several attempts of answers have been proposed in [62, 43]. Each of these works deals with a model where the magnetic field is not exactly parallel but grazing, that is the angle is $\theta = \frac{\pi}{2} - \eta$ where η is a small angle. Both works [62, 43] agree with the fact that decreasing the magnetic field intensity results in a rescaling of the Chodura sheath, whose thickness increases. At the same time the charge separation near the wall decreases with decreasing magnetic field intensity and almost disappear for very small magnetic field intensity. The physical interpretation given by Manfredi and Coulette is that decreasing the magnetic field results in a Chodura sheath whose thickness is way larger than the thickness of the Debye sheath, the whole potential drop needed to confine the plasma occurs more easily in the Chodura sheath with hardly any need for a non neutral Debye sheath. Surprisingly, these observations seem to be in contradiction with the results obtained in chapter 2. Indeed, in the absence of magnetic field, there is a net charge separation at the wall. It thus questions the extension of the analysis in [43] to the case of a vanishing magnetic field. On the other side, when the intensity of the magnetic field increases the charge separation near the wall tends to increase and the ion flux perpendicular to the wall tends to deacrease. This last physical effect being interpreted in [43] as the formation of a new non neutral sheath due to the fact that the thickness of the Chodura sheath and the Debye sheath becomes of the same order. The two sheath overlaps.

As far as the case of a parallel magnetic field is concerned, Stangeby in his book [63] mentions that the physics is for the moment not completely understood, and that producing an analysis in that case still is an open problem. However, a recent work [48] investigates numerically the case of a parallel to the wall magnetic field. It is observed that for an increase of the ratio ω_{ci}/ω_{pi} between the plasma ion gyrofrequency and the plasma ion frequency (that is relatively large), the potential drop decreases and the extent of the non neutral area increases. In this work, we propose to study a model of plasma wall interaction with a self consistent potential and a constant magnetic field that is exactly parallel to the wall. It is found that a parallel magnetic field simplifies the mathematial analysis because three invariants allow for an exact representation solutions of the Vlasov equation. Ions and electrons densities are solutions to four dimensional Vlasov equations coupled with the Poisson equation. Boundary conditions are determined to reflect the physical properties of this simplified model: in particular, at the wall we consider that electrons are re-emitted with probability α while ions are totally absorbed. Ions and electrons are assumed to enter the plasma with given velocity distributions. Since the core of the plasma is well described by a full Maxwellian, we have chosen to consider Maxwellian distributions for the incoming electrons. The wall potential is determined so that the ambipolarity holds for the stationary solutions. The resulting potential is well-defined for an electric field that satisfies a lower bound on its derivative, and for incoming ions satisfying two conditions:

- their v_z velocities is negative and lower than some bound that depends on the intensity of the magnetic field,
- an upper bound on their average v_x velocity.

Because we want to avoid the existence of a trivial solution, the first condition turns out to prevent ion particles to be naturally confined by the magnetic field even with a weak electric-field. Incoming ions particles in our model are fated to reach the wall. The second condition limits the ion flux perpendicular to the wall and is reminiscent of the condition identified in chapter 2 for the existence of a negative wall potential. At the numerical level, we observe two things : on the one hand increasing the magnetic field results in a stronger limitation of the ion flux perpendicular the wall and thus the negative wall potential cannot be determined for too large magnetic field. On the other hand, increasing the magnetic field results in a wall potential (in absolute value) that diminishes and a positive charge at the wall that increases. These results are in good agreement with the analysis made by Manfredi and Coulette in [43] for a magnetic field with a low incidence angle. In the simulations we present, the ion Larmor radius is way larger than the Debye length, this is typically the case for Tokamak plasmas. Other scalings are not considered.

The plan of this chapter is as follows. In section 4.2 we begin with a presentation of the model and write down its mathematical structure. Then in section 4.3.1, we show how to reduce the Vlasov-Poisson system to a non linear Poisson equation. In section 4.4, we prove under the assumptions above described that we can determine a priori the wall potential by solving a non linear equation. As a result, a necessary and sufficient condition for the wall potential to be uniquely determined is established. This condition takes the form of an upper bound on the ion flux perpendicular to the wall. The resulting potential does not depend on the Debye length. In section 4.5 we set up the mathematical framework that is used to study the non linear Poisson equation. We next study the non linear Poisson equation as a minimization problem. Finally, in section 4.6, we describe the numerical methods employed to solve the problem and present some numerical results. Final comments about the range of applicability of this work are provided as a conclusion in section 4.7.

4.2 Description of the bi-kinetic model

4.2.1 Physical setting

We consider a plasma at equilibrium made of one species of ions and electrons. This plasma is assumed to be contained in a one dimensional chamber. This model only describes a portion of the chamber of length $L > 0$. Physical quantities will often be denoted with upper case while normalized ones will be denoted with lower case. Our system is subject to the following physical considerations:

1. The plasma is non collisional and we neglect the effect of the self-consistent magnetic field.
2. The plasma is subject to an external magnetic field that is constant in space and time, and parallel to the wall, that is of the form
$$\mathbf{B} := \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix} \text{ where } B > 0 \text{ is the amplitude.} \tag{4.3}$$
3. We assume the invariance on (Y, Z) of the physical quantities that describe the plasma state such as the electrostatic potential, the ionic distribution and the electronic distribution (that we will denote Φ, F_i and F_e).
4. At $X = 0$, we consider:
   a) that the potential Φ is arbitrarily set to zero;
   b) that ions and electrons are injected with positive orthogonal to the wall velocity component described through velocity distributions denoted respectively F_i^{in} and F_e^{in};
   c) the plasma is neutral, i.e, there is an equal number of ions and electrons.
5. At the wall, that is at $X = L$, we consider:
   a) purely absorbing conditions for ions, i.e, ions are not re-emitted from the wall;
   b) electrons re-emitted with probability $\alpha \in [0, 1]$;
   c) as in [43] that there is no net current in the orthogonal direction to the wall.

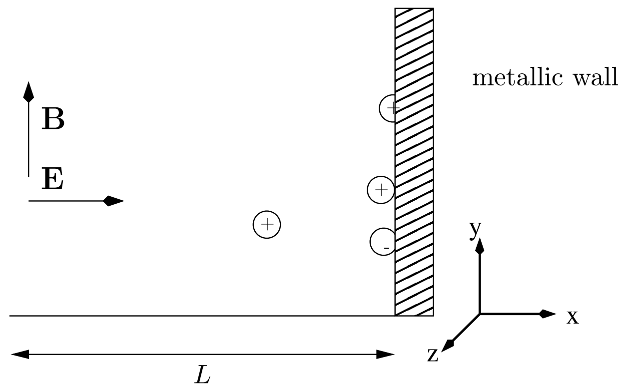


Figure 4.1 – Ions and electrons are injected in the direction to the wall. Some electrons reaching the wall are re-emitted with a probability $\alpha \in [0,1]$ while ions are totally absorbed. The electric field is self-consistent and the magnetic field is parallel to the wall and constant in space and time.

4.2.2 Kinetic modeling of the stationary plasma wall interaction

We first write the equations in physical units and then derive a dimensionless model. The space variable is denoted $X \in [0,L]$ and the velocity variable is denoted $\mathbf{V} := (V_x, V_y, V_z)^t \in \mathbb{R}^3$. In this model, the unknowns function are the electrostatic potential $\Phi : [0,L] \to \mathbb{R}$, the ion distribution function $F_i : [0,L] \times \mathbb{R}^3 \to \mathbb{R}^+$, the electron distribution function $F_e : [0,L] \times \mathbb{R}^3 \to \mathbb{R}^+$. We shall also consider two degrees of freedom: the reference density $N_0 > 0$ that will be a parameter of the incoming electron boundary condition and the wall potential $\Phi_w \in \mathbb{R}$ that is a boundary condition for the electric potential Φ. We denote by $N_i(X) := \int_{\mathbb{R}^3} F_i(X,\mathbf{V})d\mathbf{V}$ (respectively $N_e(X) := \int_{\mathbb{R}^3} F_e(X,\mathbf{V})d\mathbf{V}$) the ionic (respectively the electronic) density at $X \in [0,L]$ and $\Gamma_i(X) := \int_{\mathbb{R}^3} F_i(X,\mathbf{V})V_x d\mathbf{V}$ (respectively $\Gamma_e(X) := \int_{\mathbb{R}^3} F_e(X,\mathbf{V})V_x d\mathbf{V}$) the ionic (respectively the electronic) flux in the orthogonal to the wall direction at $X \in [0,L]$. The equations governing the ion and electron transport in the plasma, with an electric field $-\partial_X \Phi$ and the magnetic-field $\mathbf{B}$ defined by (4.3) are assumed to be stationary Vlasov equations and write

$$V_x \partial_X F_i + \frac{q}{m_i}\left(-\partial_X \Phi \mathbf{e_x} + \mathbf{V} \times \mathbf{B}\right).\nabla_V F_i = 0 \quad \forall (X,\mathbf{V}) \in (0,L) \times \mathbb{R}^3, \tag{4.4}$$

$$V_x \partial_X F_e - \frac{q}{m_e}\left(-\partial_X \Phi \mathbf{e_x} + \mathbf{V} \times \mathbf{B}\right).\nabla_V F_e = 0 \quad \forall (X,\mathbf{V}) \in (0,L) \times \mathbb{R}^3, \tag{4.5}$$

with the boundary conditions

$$F_e(0,\mathbf{V}) = N_0 F_e^{in}(\mathbf{V}) \quad \forall \mathbf{V} \in (0,+\infty) \times \mathbb{R}^2, \tag{4.6}$$

$$F_e(L,\mathbf{V}) = \alpha F_e(L,-V_x,V_y,V_z) \quad \forall \mathbf{V} \in (-\infty,0) \times \mathbb{R}^2, \tag{4.7}$$

$$F_i(0,\mathbf{V}) = F_i^{in}(\mathbf{V}) \quad \forall \mathbf{V} \in (0,+\infty) \times \mathbb{R}^2, \tag{4.8}$$

$$F_i(L,\mathbf{V}) = 0 \quad \forall \mathbf{V} \in (-\infty,0) \times \mathbb{R}^2, \tag{4.9}$$

where $q > 0$ is the electric charge and m_i (respectively m_e) denotes the ionic (respectively the electronic) mass. A formal integration of equation (4.4)-(4.5) with respect to velocity variables shows that the current density in the orthogonal direction to the wall, defined for all $X \in [0, L]$ by $J(X) := q\Gamma_i(X) - q\Gamma_e(X)$ is constant in space and so $J(X) = J(0) = J(L)$ for all $X \in [0, L]$. According to our physical considerations we therefore require that

$$J(X) = 0 \quad \forall X \in [0, L]. \tag{4.10}$$

The electric potential is determined through the Gauss law

$$-\frac{d^2}{dX^2}\Phi(X) = \frac{q}{\varepsilon_0}(N_i(X) - N_e(X)) \quad \forall X \in (0, L) \tag{4.11}$$

with boundary conditions

$$\Phi(0) = 0, \quad \Phi(L) = \Phi_w, \tag{4.12}$$

where ε_0 is the vacuum permittivity. Lastly, the equation of neutrality at $X = 0$ writes

$$N_i(0) = N_e(0). \tag{4.13}$$

For the mathematical analysis of this model, it is convenient to rescale equations and to this end we introduce the dimensionless variables, x, v and the dimensionless functions ϕ, f_i and f_e defines as:

$$x := \frac{X}{L}, \quad \mathbf{v} := \frac{\mathbf{V}}{c_s},$$

$$f_i(x, \mathbf{v}) := Lc_s^3 F_i(Lx, c_s\mathbf{v}), \quad f_e(x, \mathbf{v}) := Lc_s^3 F_e(Lx, c_s\mathbf{v}), \quad \phi(x) := \frac{q}{k_B T_e}\Phi(Lx),$$

where k_B is the Boltzmann constant, T_e is a reference electron temperature, and $c_s := \sqrt{\frac{k_B T_e}{m_i}}$ denotes the ion acoustic velocity. We then define the dimensionless quantities:

$$n_i(x) := \int_{\mathbb{R}^3} f_i(x, \mathbf{v})d\mathbf{v}, \quad n_e(x) := \int_{\mathbb{R}^3} f_e(x, \mathbf{v})d\mathbf{v},$$

$$\gamma_i(x) := \int_{\mathbb{R}^3} f_i(x, \mathbf{v})v_x d\mathbf{v}, \quad \gamma_e(x) := \int_{\mathbb{R}^3} f_e(x, \mathbf{v})v_x d\mathbf{v},$$

$$n_0 := LN_0, \quad \phi_w := \frac{q}{k_B T_e}\Phi_w,$$

$$f_e^{in}(v) = c_s^3 F_e^{in}(Lv), \quad f_i^{in}(v) = Lc_s^3 F_i^{in}(Lv).$$

We also introduce the dimensionless parameters

$$\mu := \frac{m_e}{m_i}, \quad \omega_i := \frac{qB}{m_i}\frac{L}{c_s}, \quad \varepsilon := \sqrt{\frac{k_B T_e \varepsilon_0}{q^2 L}},$$

that denote respectively the mass ratio, the normalized ion gyrofrequency and the normalized Debye length. The coupled boundary value problem (4.4)-(4.13) is then equivalent to the following boundary value problem:

$$\begin{cases} v_x \partial_x f_i + \left(-\partial_x \phi \mathbf{e_x} + \omega_i \mathbf{v} \times \mathbf{b}\right).\nabla_v f_i = 0 \quad \forall (x, \mathbf{v}) \in (0,1) \times \mathbb{R}^3, & (4.14) \\ v_x \partial_x f_e - \dfrac{1}{\mu}\left(-\partial_x \phi \mathbf{e_x} + \omega_i \mathbf{v} \times \mathbf{b}\right).\nabla_v f_e = 0 \quad \forall (x, \mathbf{v}) \in (0,1) \times \mathbb{R}^3, & (4.15) \\ -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = n_i(x) - n_e(x), \quad \forall x \in (0,1) & (4.16) \\ \gamma_i(x) - \gamma_e(x) := j(x) = 0, \quad \forall x \in [0,1], & (4.17) \\ n_i(0) - n_e(0) = 0. & (4.18) \end{cases}$$

complemented with the boundary conditions

$$\begin{cases} f_e(0, \mathbf{v}) = n_0 f_e^{in}(\mathbf{v}) \, \forall \mathbf{v} \in (0, +\infty) \times \mathbb{R}^2, \; f_e(1, \mathbf{v}) = \alpha f_e(1, -v_x, v_y, v_z) \, \forall \mathbf{v} \in (-\infty, 0) \times \mathbb{R}^2 & (4.19) \\ f_i(0, \mathbf{v}) = f_i^{in}(\mathbf{v}) \quad \forall \mathbf{v} \in (0, +\infty) \times \mathbb{R}^2, \quad f_i(1, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in (-\infty, 0) \times \mathbb{R}^2, & (4.20) \\ \phi(0) = 0, \quad \phi(1) = \phi_w. & (4.21) \end{cases}$$

In this model, we have five unknowns ϕ, f_i, f_e, n_0 and ϕ_w and five equations (4.14)-(4.18). The unknowns ϕ, f_i, f_e and ϕ_w can be seen as state-variables of our physical system, in the sense that they are determined by fundamental physical laws. The unknown n_0 is more like a parameter to be adjusted to ensure the neutrality at $x = 0$. The neutrality is not, properly said, fundamental and its status is questionable. The introduction of the unknown n_0 enables us to get rid of a possible constrain on the incoming boundary conditions. This bi-kinetic boundary value problem is inspired from [43] where only the ions are described with a kinetic equation, the electrons are assumed to be Boltzmannian. At a mathematical level, we are not aware of works dealing exactly with this problem, however we mention the work of Raviart and Greengard [54] where the authors construct stationary solutions to a one dimensional Vlasov-Poisson system in a bounded domain. Their technique employs a priori L^∞ estimate of the charge combined with a fixed point argument. In the higher dimensional case and still in bounded domains, there is the work [1] that deals with stationary solutions of a Vlasov-Poisson system in bounded domains. The existence of stationary solutions for the Vlasov-Maxwell system in bounded domains was established by Poupaud in [53]. More in the spirit of this work, the recent work [37] studies the stationary solution to a Vlasov-Poisson system. The technique employed to establish the existence of stationary solutions in [1, 37] consists in writing the solution as the critical point of some energy functional. Our approach clearly follows the same idea, especially we are going to present a minimization formulation of our Vlasov-Poisson system. In a previous work of our [7] on a one dimensional bi-kinetic sheath model, we have already used minimization techniques.

4.3 Reformulation as a non linear Poisson problem

We are going to reduce the Vlasov-Poisson problem (4.15), (4.14), (4.16) and (4.19)-(4.21) to a non linear Poisson problem. When the potential ϕ is given both Vlasov equations for ions and electrons are linear transport equations, and their solutions are determined by transport along the characteristics of the (incoming) boundary conditions. We assume the following properties:

$$\begin{gathered} \forall\, s = i, e, f_s^{in} \in (L^1 \cap L^\infty)(\mathbb{R}^+ \times \mathbb{R}^2; \mathbb{R}^+) \text{ with} \\ \int_{\mathbb{R}^2} \int_0^{+\infty} f_s^{in}(v_x, v_y, v_z) v_x^2 dv_x dv_y dv_z < +\infty, \end{gathered} \tag{4.22}$$

$$\phi \in W^{2,\infty}(0,1),\ \phi'' \leqslant \frac{\omega_i^2}{\mu} \text{ almost everywhere in } (0,1) \text{ and for all } x \in [0,1]\ \phi(x) \leqslant 0, \tag{4.23}$$

$$\text{for almost every } (v_x, v_y, v_z) \in (0,+\infty) \times \mathbb{R} \times [-\frac{\omega_i}{2}, +\infty)\ f_i^{in}(v_x, v_y, v_z) = 0. \tag{4.24}$$

We shall see more in details in section 4.3.1 to which extent the assumptions (4.22)-(4.24) are, if not necessary, sufficient and useful in order to simplify at first glance the study of the Vlasov-Poisson system (4.14)-(4.21). Let us, however, make some comments on these assumptions:

- The assumption (4.23) is reminiscent of the one made in [20]. It results from the convexity of some microscopic "electron potential" that takes into account the magnetic part and the electric part. It is useful to make the current electron density γ_e and the electron density at $x = 0$, $n_e(0)$ independent of the potential ϕ. The assumption (4.24) combined with the sign condition $\phi \leqslant 0$ are sufficient conditions for some microscopic "ion potential" to take its maximum value at $x = 0$ whenever there is particles with v_z velocities upper bounded by $-\frac{\omega_i}{2}$. Similarly to the electrons, this assumption makes the ion current density γ_i and the ion density at $x = 0$ to be independent of ϕ. Ultimately, these two assumptions (4.23) and (4.24) enable to decouple (from the full Vlasov-Poisson system) the ambipolarity equation (4.17) and the neutrality relation (4.18) that will concern only n_0 and ϕ_w.

- The assumption on the support of f_i^{in} could probably be relaxed. Typically, an assumption on the convexity of U_i, namely $\phi'' \geqslant -\omega_i^2$ almost every where in $(0,1)$ would enable one to get rid of the support condition for f_i^{in}. A reason why we have not used it here, is that combined with the previous assumption $\phi'' \leqslant \frac{\omega_i^2}{\mu}$ we get in the limit $\omega_i \to 0^+$ that $\phi'' = 0$ which seems to be unsatisfactory. Our previous work in the absence of magnetic field clearly showed that $\phi'' < 0$ near $x = 1$ (the wall). We hope this model to be able to capture the "reference"solution even when ω_i is small.

The main result of this section is the following.

Proposition 4.3.1 (Reduction)**.** *Let $n_0 > 0$, $\phi_w < 0$. Assume (4.22)-(4.24) with $\phi(0) = 0$ and $\phi(1) = \phi_w$. Then there exists $(f_i, f_e) \in L^\infty((0,1) \times \mathbb{R})^2$ with f_e that vanishes on closed characteristics and such that the Vlasov-Poisson system holds if and only if ϕ is a solution to*

$$\text{(NLP-MAG)} : \begin{cases} -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = (n_i - n_e)(x) & \text{for a.e } x \in (0,1) \\ \phi(0) = 0, \quad \phi(1) = \phi_w \end{cases} \tag{4.25}$$

with

$$n_i(x) = \int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} \frac{f_i^{in}(w_x, I_2, I_3) w_x}{\sqrt{w_x^2 + 2\delta_i^0(x, I_3)}} d_w dI_3 dI_2,$$

$$
\begin{aligned}
n_e(x) =& n_0(1+\alpha)\int_{\mathbb{R}}\int_{-\infty}^{E(\phi_w)}\int_{\sqrt{2\delta_e^1(0,E_3)}}^{+\infty}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_x dE_3 dE_2\\
& n_0(1+\alpha)\int_{\mathbb{R}}\int_{E(\phi_w)}^{+\infty}\int_{0}^{+\infty}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_x dE_3 dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)\geqslant E(\phi_w)}(x)\int_{\mathbb{R}}\int_{-\infty}^{E(\phi_w)}\int_{\sqrt{-2\delta_e^0(x,E_3)}}^{\sqrt{2\delta_e^1(0,E_3)}}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_x dE_3 dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)< E(\phi_w)}(x)\int_{\mathbb{R}}\int_{-\infty}^{\underline{E}(x)}\int_{\sqrt{-2\delta_e^0(x,E_3)}}^{\sqrt{2\delta_e^1(0,E_3)}}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_x dE_3 dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)< E(\phi_w)}(x)\int_{\mathbb{R}}\int_{\underline{E}(x)}^{E(\phi_w)}\int_{0}^{\sqrt{2\delta_e^1(0,E_3)}}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_x dE_3 dE_2
\end{aligned}
$$

where the function $\underline{E}$ is defined for all $x\in(0,1]$ by $\underline{E}(x) := \frac{1}{\omega_i x}\left(\frac{\omega_i^2 x^2}{2\mu}-\phi(x)\right)$ and extended by continuity at $x=0$ with $\underline{E}(0) = -\frac{\phi'(0)}{\omega_i}$, $E(\phi_w)=\underline{E}(1)$ and where $\delta^0(x,E_3)$ will be made precise later.

The proof of the above proposition relies on the following ingredients:

a) The conservation of three invariants along the characteristic curves.

b) The explicit representation of the distribution functions.

4.3.1 Study of the linear Vlasov system: construction of the invariants and representation of the distributions

When the potential ϕ is given, both Vlasov equations (4.14),(4.15) are linear transport equations and their solutions are determined by transport along the characteristics of the (incoming) boundary conditions. In this section, we assume to be known $n_0>0$, $\phi_w<0$ and $\phi\in W^{2,\infty}(0,1)$ such that $\phi(0)=0$ and $\phi(1)=\phi_w$. This regularity of ϕ is sufficient to define the characteristic curves [23]. We also assume (4.22)-(4.24).

Study of the ion characteristics and invariants

Definition 4.3.2. *The characteristic trajectories of the ions are the curves which satisfy the ordinary differential system of equations*

$$
(C_i):\begin{cases}
\dot{\mathcal{X}}(t)=\mathcal{V}_x(t),\\
\dot{\mathcal{V}}_x(t)=-\phi'(\mathcal{X}(t))-\omega_i\mathcal{V}_z(t),\\
\dot{\mathcal{V}}_y(t)=0,\\
\dot{\mathcal{V}}_z(t)=\omega_i\mathcal{V}_x(t),\\
\mathcal{X}(0)=x\in[0,1],\\
\mathcal{V}(0)=\mathbf{v}\in\mathbb{R}^3.
\end{cases}
$$

For any arbitrary initial data $(x,\mathbf{v})\in(0,1)\times\mathbb{R}^3\cup\big(\{0\}\times[0,+\infty)\times\mathbb{R}^2\big)\cup\big(\{1\}\times(-\infty,0]\times\mathbb{R}^2\big)$ there is a unique solution denoted $(\mathcal{X}(t;x,\mathbf{v}),\mathcal{V}(t;x,\mathbf{v}))$ for all $t\in[t_{in}(x,\mathbf{v}),t_{out}(x,\mathbf{v})]$ where

$$
t_{in}(x,\mathbf{v}) := \inf\{\tau\leqslant 0 : \mathcal{X}(s,x,\mathbf{v})\in(0,1),\forall s\in(\tau,0)\},
$$

$$t_{out}(x,\mathbf{v}) := \sup\{\tau \geqslant 0, \mathcal{X}(s,x,\mathbf{v}) \in (0,1), \forall s \in (0,\tau)\}.$$

We have three constant of motions for the system (C_i). Indeed, let us introduce the three invariant functions defined for all $x \in [0,1]$ and $\mathbf{v} \in \mathbb{R}^3$ by

$$\begin{cases} I_1(x,\mathbf{v}) := \dfrac{1}{2}\left(v_x^2+v_y^2+v_z^2\right)+\phi(x), & (4.26)\\ I_2(x,\mathbf{v}) := v_y, & (4.27)\\ I_3(x,\mathbf{v}) := v_z-\omega_i x. & (4.28)\end{cases}$$

Lemma 4.3.3. *The functions I_1,I_2 and I_3 are constant along the characteristics.*

Proof. For all $k \in \{1,2,3\}$, the function $c_k : t \in [t_{in}(x,\mathbf{v}), t_{out}(x,\mathbf{v})] \mapsto I_k(\mathcal{X}(t;x,\mathbf{v}), \mathcal{V}(t;x,\mathbf{v}))$ is differentiable on $(t_{in}(x,\mathbf{v}), t_{out}(x,\mathbf{v}))$ with $c_k' = 0$ thus c_k is constant. □

Thanks to Lemma 4.3.3, each characteristic curve is included in a set $\mathcal{C}_{I_1,I_2,I_3}$ defined as the points $(x,y,v_x,v_y,v_z) \in [0,1]\times\mathbb{R}^3$ satisfying the system

$$\begin{cases} I_1 = \frac{1}{2}\left(v_x^2+v_y^2+v_z^2\right)+\phi(x),\\ I_2 = v_y,\\ I_3 = v_z-\omega_i x.\end{cases}$$

where $(I_1,I_2,I_3) \in \mathbb{R}^3$ are now parameters. To study the set $\mathcal{C}_{I_1,I_2,I_3}$, it is convenient to define following function:

$$\forall I_3 \in \mathbb{R}, \forall x \in [0,1] \quad U_i(x,I_3) := \frac{(I_3+\omega_i x)^2}{2}+\phi(x). \tag{4.29}$$

Then we have for all $(I_1,I_2,I_3) \in \mathbb{R}^3$,$(x,v_x,v_y,v_z) \in \mathcal{C}_{I_1,I_2,I_3}$ if and only if:

$$\begin{cases} \frac{v_x^2}{2} = \mathcal{I} - U_i(x,I_3),\\ v_y = I_2,\\ v_z = I_3+\omega_i x,\\ \mathcal{I} = I_1 - \frac{I_2^2}{2}.\end{cases}$$

The set $\mathcal{C}_{I_1,I_2,I_3}$ is not empty if and only if there is $x \in [0,1]$ such that $U_i(x,I_3) \leqslant \mathcal{I}$. When it is non-empty the set $\mathcal{C}_{I_1,I_2,I_3}$ consists of one or several curves in the four dimensional phase space $[0,1]\times\mathbb{R}^3$. In order to have a better representation of these curves, we introduce for each $I_3 \in \mathbb{R}$ the maximum of the function $U_i(.,I_3)$ namely:

$$\forall I_3 \in \mathbb{R}, \quad U_i^M(I_3) := \max_{x\in[0,1]} U_i(x,I_3).$$

It is well defined since the function $U_i(.,I_3)$ is continuous on $[0,1]$. We shall also need to split the physical space according to the locations of maxima in the interval [0,1]. We therefore introduce

$$x_M^-(I_3) := \min\{x \in [0,1] \mid U_i(x,I_3) = U_i^M(I_3)\}$$

as well as the two following sets defined for $I_3 \in \mathbb{R}$ by

$$D_{I_3} := \left\{(x,v_x) \in [0,1]\times\mathbb{R} \ \text{ s.t } |v_x| \geqslant \sqrt{2(U_i^M(I_3)-U_i(x,I_3))}\right\},$$

$$S_{I_3} := \left\{ (x, v_x) \in [0,1] \times \mathbb{R} \text{ s.t } |v_x| = \sqrt{2(U_i^M(I_3) - U_i(x, I_3))} \right\}.$$

One has the decomposition $[0,1] \times \mathbb{R} = D_{I_3} \cup D_{I_3}^c$ for all $I_3 \in \mathbb{R}$. The set S_{I_3} is the separatrix curve of the (x, v_x) plane, it splits the (x, v_x) plane into two zones, namely:

a) these are the points of D_{I_3}: that are on a curve $\mathcal{C}_{I_1,I_2,I_3}$ that crosses the two boundaries $x = 0$ and $x = 1$.

b) these are the points of $D_{I_3}^c$: that are on a curve $\mathcal{C}_{I_1,I_2,I_3}$ that do not cross the two boundaries $x = 0$ and $x = 1$, i.e, either they cross only one boundary or they never intersect the boundaries.

We decompose the four dimensional phase space as follows,

$$[0,1] \times \mathbb{R}^3 = \bigcup_{(I_2,I_3)\in\mathbb{R}^2} \left\{ (x, v_x, v_y, v_z) \in [0,1] \times \mathbb{R}^3 \text{ s.t } (x, v_x) \in D_{I_3} \cup D_{I_3}^c,\ v_y = I_2,\ v_z = I_3 + \omega_i x \right\}.$$

Remark 17. *The family of sets defined for each $(I_2, I_3) \in \mathbb{R}^2$ by*

$$Q_{I_2,I_3} := \left\{ (x, v_x, v_y, v_z) \in [0,1] \times \mathbb{R}^3 \ s.t\ (x, v_x) \in D_{I_3} \cup D_{I_3}^c,\ v_y = I_2,\ v_z = I_3 + \omega_i x \right\}$$

is a partition of the phase space $[0,1] \times \mathbb{R}^3$. Each of these sets is the intersection of the portion of two orthogonal hyperplanes, namely:

$$\mathcal{P}_{I_2} \text{ is the hyperplane with normal vector } (0,0,1,0)^t \text{ passing through } (0,0,I_2,0)^t,$$

and

$$\mathcal{P}_{I_3} \text{ is the hyperplane with normal vector } (-\omega_i,0,0,1)^t \text{ passing through } (0,0,0,I_3)^t.$$

Remark 18. *With this partitioning of the phase space, we know wheter a point $(x, v_x, v_y, v_z) \in [0,1] \times \mathbb{R}^3$ is originally on a characteristic curve that reaches the two boundaries or not. Indeed, it suffices to check if (x, v_x) belongs to D_{I_3} where $I_3 = v_z - \omega_i x$.*

Remark 19. *Generally, it is possible that there are closed characteristic curves that do not cross the boundaries. If so, the solution to the linear Vlasov equation is not unique because on these curves the solution is not determined by the (incoming) boundary conditions, it can take arbitrary values. In this work, thanks to the assumptions (4.23)-(4.24) this cannot happen.*

Let us give an illustration of the partition of the phase space in Figure 4.2

Remark 20. *Thanks to the assumptions (4.24) and (4.22), our study will be limited to the case $I_3 < -\frac{\omega_i}{2}$. In this configuartion, it happens that $x_M^-(I_3) = 0$ and the graph of the phase space in Figure 4.2 reduces to the part of graph where $x \geqslant x_M^-(I_3)$.*

To finish with this section, let us introduce the following objects that are meant to facilitate the reading of the following section dedicated to the construction of a solution. We defined for all $x \in [0,1]$ and $I_3 \in \mathbb{R}$,

$$\delta_i^0(x, I_3) := U_i(0, I_3) - U_i(x, I_3), \quad \delta_i^1(x, I_3) := U_i(1, I_3) - U_i(x, I_3), \quad \delta_i^M(x, I_3) := U_i^M(I_3) - U_i(x, I_3) \geqslant 0,$$

$$V^M(x, I_3) := \{ v_x \in \mathbb{R} \,|\, |v_x| \geqslant \sqrt{2\delta_i^M(x, I_3)} \}, \quad V^0(x, I_3) := \{ v_x \in \mathbb{R} \,|\, v_x^2 \geqslant 2\delta_i^0(x, I_3) \},$$

$$V^1(x, I_3) := \{ v_x \in \mathbb{R} \,|\, v_x^2 \geqslant 2\delta_i^1(x, I_3) \}.$$

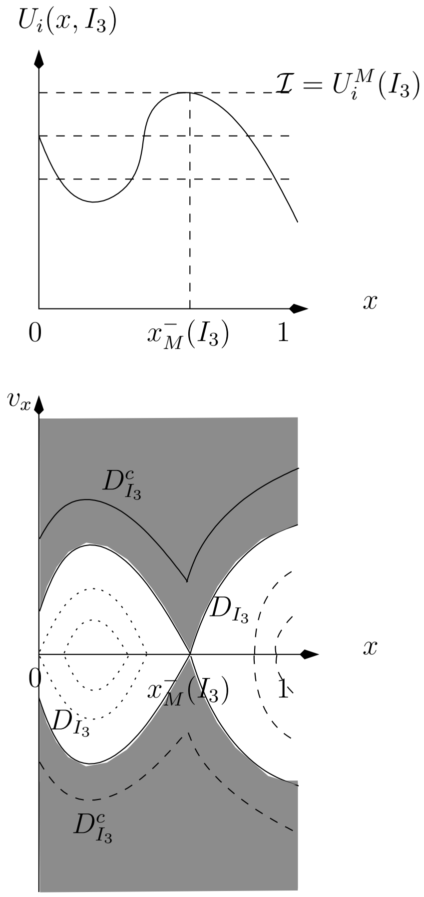


Figure 4.2 – Sketch of the phase-space (x, v_x) associated with the potential $U_i(., I_3)$ defined by (4.29) and for a given value $I_3 \in \mathbb{R}$ and where $I_1 \in \mathbb{R}$ and $I_2 \in \mathbb{R}$ are such that $\mathcal{I} = I_1 - \frac{I_2^2}{2}$. Dashed lines correspond to characteristics that originate at $x = 1$ with negative velocities. Because of the boundary condition, no particles travel on these curves. Bold lines correspond to characteristics that originate at $x = 0$ with positive velocites. Dotted lines corresponds to closed characteristics, on these curves the distribution function can take arbitrary values.

Construction of a solution for the ions

Because of the boundary conditions and the geometry of the characteristics, we shall consider weak solutions of the Vlasov equations.

Definition 4.3.4 (Weak solution)**.** *Let $v_x f_i^{in} \in L^1_{loc}([0, +\infty) \times \mathbb{R}^2)$. We say that $f_i \in L^1_{loc}([0, 1] \times \mathbb{R}^3)$ is a weak solution of the Vlasov equation (4.14) iff for all $\varphi \in C^1_c([0, 1] \times \mathbb{R}^3)$ with $\varphi(1, \mathbf{v}) = 0$ for $v_x \geqslant 0$ and $\varphi(0, \mathbf{v}) = 0$ for $v_x \leqslant 0$ we have*

$$-\int_0^1 \int_{\mathbb{R}^3} f_i(x, \mathbf{v})\psi(x, \mathbf{v}) dx d\mathbf{v} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^+} f_i^{in}(\mathbf{v})\varphi(0, \mathbf{v}) v_x dv_x dv_y dv_z,$$

where $\psi(x, \mathbf{v}) := v_x \partial_x \varphi(x, \mathbf{v}) + (-\partial_x \phi(x) \mathbf{e_x} + \omega_i \mathbf{v} \times \mathbf{b}) . \nabla_v \varphi(x, \mathbf{v})$.

Using the fact that solutions of the linear Vlasov equation are constant along the characteristics, we are going to define a solution. We define the function f_i as follows: For all $(x, v_x, v_y, v_z) \in [0,1] \times \mathbb{R}^3$, let $I_2 = v_y$ and $I_3 = v_z - \omega_i x$ then we define

$$f_i(x, v_x, v_y, v_z) = \begin{cases} \text{If } (x, v_x) \in D_{I_3} \setminus S_{I_3} \text{ then} \\ f_i^{in}(\sqrt{v_x^2 - 2\delta_i^0(x, I_3)}, I_2, I_3) \text{ for } v_x > 0, \\ 0 \text{ for } v_x < 0, \\ \text{If } (x, v_x) \in D_{I_3}^c \text{ then} \\ f_i^{in}(\sqrt{v_x^2 - 2\delta_i^0(x, I_3)}, I_2, I_3) \text{ if } (x, v_x) \in [0, x_M^-(I_3)] \times V^0(x, I_3), \\ 0 \text{ if } (x, v_x) \notin [0, x_M^-(I_3)] \times V^0(x, I_3). \end{cases} \tag{4.30}$$

Remark 21. *The set* $\bigcup_{(I_2,I_3)\in\mathbb{R}^2} \{(x, v_x, v_y, v_z) \in [0,1] \times \mathbb{R}^3 \text{ s.t } (x, v_x) \in S_{I_3}, v_y = I_2, v_z = I_3 + \omega_i x\}$ *being of measure zero (because it is a three dimensional volume) we have not defined* f_i *on this set. This is not necessary since we consider weak solutions that are functions almost everywhere defined.*

One can prove the following.

Proposition 4.3.5. *The function* f_i *defined by* (4.30) *is a weak solution of the Vlasov equation.*

First three moments of the ion distribution

Definition 4.3.6. *We define the density, the current density and the kinetic energy associated to the ion distribution* f_i *defined in* (4.30) *by the functions defined respectively for all* $x \in [0,1]$ *by*

$$n_i(x) := \int_{\mathbb{R}} f_i(x, \mathbf{v}) d\mathbf{v}, \quad \gamma_i(x) := \int_{\mathbb{R}} f_i(x, \mathbf{v}) v_x d\mathbf{v}, \quad \mathcal{E}_i(x) := \frac{1}{2} \int_{\mathbb{R}} f_i(x, \mathbf{v}) v_x^2 d\mathbf{v}.$$

Using the definition of f_i and straightforward integration yields the following proposition.

Proposition 4.3.7. *(Ion density) For all* $x \in [0,1]$ *we have*

$$\begin{aligned} n_i(x) = &\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\sqrt{2\delta_i^M(x,I_3)}}^{+\infty} f_i^{in}(\sqrt{v_x^2 - 2\delta_i^0(x, I_3)}, I_2, I_3) dv_x dI_3 dI_2 \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{|v_x| < \sqrt{2\delta_i^M(x,I_3)}\} \cap \{v_x^2 \geqslant 2\delta_i^0(x,I_3)\}} \mathbf{1}_{[0, x_M^-(I_3)]}(x) f_i^{in}(\sqrt{v_x^2 - 2\delta_i^0(x, I_3)}, I_2, I_3) dv_x dI_3 dI_2. \end{aligned}$$

We can also compute the current and the kinetic energy.

Proposition 4.3.8. *(Ion current density) For all* $x \in [0,1]$

$$\gamma_i(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\sqrt{2\delta_i^M(0,I_3)}}^{+\infty} f_i^{in}(w_x, I_2, I_3) w_x dw_x dI_2 dI_3.$$

Proof. Using the definition of f_i we get that for all $x \in [0,1]$

$$\begin{aligned} \gamma_i(x) = &\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\sqrt{2\delta_i^M(x,I_3)}}^{+\infty} f_i^{in}(\sqrt{v_x^2 - 2\delta_i^0(x, I_3)}, I_2, I_3) v_x dv_x dI_3 dI_2 \\ &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{|v_x| < \sqrt{2\delta_i^M(x,I_3)}\} \cap \{v_x^2 \geqslant 2\delta_i^0(x,I_3)\}} \mathbf{1}_{[0, x_M^-(I_3)]}(x) f_i^{in}(\sqrt{v_x^2 - 2\delta_i^0(x, I_3)}, I_2, I_3) v_x dv_x dI_3 dI_2. \end{aligned}$$

By symmetry the second integral vanishes. □

Proposition 4.3.9. *(Ion kinetic energy) For all $x \in [0,1]$*

$$2\mathcal{E}_i(x) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\sqrt{2\delta_i^M(x,I_3)}}^{+\infty} f_i^{in}(\sqrt{v_x^2 - 2\delta_i^0(x,I_3)}, I_2, I_3) v_x^2 dv_x dI_3 dI_2$$
$$+ \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\{|v_x|<\sqrt{2\delta_i^M(x,I_3)}\}\cap\{v_x^2 \geqslant 2\delta_i^0(x,I_3)\}} \mathbf{1}_{[0,x_M^-(I_3)]}(x) f_i^{in}(\sqrt{v_x^2 - 2\delta_i^0(x,I_3)}, I_2, I_3) v_x^2 dv_x dI_3 dI_2.$$

Let us now clarify why the assumptions 4.23-4.24 will be useful:

- We see the dependence on ϕ of the expressions for the density, the current and the kinetic energy through the number $x_M^-(I_3)$, $U_i^M(I_3)$ and the function δ_i^0.
- The numbers $x_M^-(I_3)$, $U_i^M(I_3)$ are not explicit. At the linear level, this does not look like a difficulty. Nevertheless at the non linear level and especially when we want to determine the wall potential ϕ_w and the density n_0, it appears to be convenient to know these numbers independently of ϕ.

Lemma 4.3.10. *Under the assumption* (4.23) *one has :*

a) $\forall I_3 < -\frac{\omega_i^2}{2}$, $U_i^M(I_3) = U_i(0, I_3)$ *and* $x_M^-(I_3) = 0$.

Proof. a). For all $I_3 < -\frac{\omega_i}{2}$ and $x \in [0,1]$ we have

$$U_i(x,I_3) - U_i(0,I_3) = \phi(x) + \frac{\omega_i^2 x^2}{2} + \omega_i I_3 x$$
$$\leqslant \omega_i x(\frac{\omega_i x}{2} + I_3) \leqslant \omega_i x(\frac{\omega_i x}{2} - \frac{\omega_i}{2}) \leqslant \frac{\omega_i^2 x}{2} x(x-1) \leqslant 0.$$

□

Remark 22. *This lemma has an interpretation in terms of physical situations we consider. For $I_3 < -\frac{\omega_i}{2}$, there is no characteristics that enter at $x = 0$ with positives v_x velocities and return at $x = 0$ with negatives v_x velocities (see figure 4.2). This clearly indicates that, in such a configuration, a particle that enters at $x = 0$ with a positive v_x velocity and with a velocity $v_z < -\frac{\omega_i}{2}$ is fated to reach $x = 1$ (the wall). It cannot be naturally confined by the magnetic field even if the electric field vanishes.*

Combined with the assumption (4.24), this lemma enables to give another expression of the density, the current and the kinetic energy where the numbers $x_M^-(I_3)$ and $U_i^M(I_3)$ no longer depend on ϕ.

Proposition 4.3.11. *Assume* (4.22)-(4.24) *then for all $x \in [0,1]$ one has:*

$$n_i(x) = \int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} \frac{f_i^{in}(w_x, I_2, I_3) w_x}{\sqrt{w_x^2 + 2\delta_i^0(x,I_3)}} d_w dI_3 dI_2, \tag{4.31}$$

$$\gamma_i(x) = \int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) w_x dw_x dI_2 dI_3, \tag{4.32}$$

$$\mathcal{E}_i(x) = \frac{1}{2}\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) w_x \sqrt{w_x^2 + 2\delta_i^0(x,I_3)} dw_x dI_3 dI_2. \tag{4.33}$$

Moreover, one has the bounds for all $x \in [0,1]$,

$$|n_i(x)| \leqslant \|f_i^{in}\|_{L^1},$$
$$|\gamma_i(x)| \leqslant \|v_x^2 f_i^{in}\|_{L^1}^{\frac{1}{2}} \|f_i^{in}\|_{L^1}^{\frac{1}{2}},$$
$$|\mathcal{E}_i(x)| \leqslant \frac{1}{2}\|v_x^2 f_i^{in}\|_{L^1}.$$

Study of the electron characteristic

Definition 4.3.12. *The characteristics trajectories of the electrons are the curves which satisfy the ordinary differential system of equations*

$$(C_e): \begin{cases} \dot{\mathcal{X}}(t) = \mathcal{V}_x(t), \\ \dot{\mathcal{V}}_x(t) = -\frac{1}{\mu}(-\phi'(\mathcal{X}(t)) - \omega_i \mathcal{V}_z(t)), \\ \dot{\mathcal{V}}_y(t) = 0, \\ \dot{\mathcal{V}}_z(t) = -\frac{\omega_i}{\mu}\mathcal{V}_x(t), \\ \mathcal{X}(0) = x \in [0,1], \\ \mathcal{V}(0) = \mathbf{v} \in \mathbb{R}^3. \end{cases}$$

For any arbitrary initial data $(x,\mathbf{v}) \in (0,1)\times\mathbb{R}^3 \cup \left(\{0\}\times[0,+\infty)\times\mathbb{R}^2\right) \cup \left(\{1\}\times(-\infty,0]\times\mathbb{R}^2\right)$ there is a unique solution denoted $(\mathcal{X}(t;x,\mathbf{v}),\mathcal{V}(t;x,\mathbf{v}))$ for all $t \in [t_{in}(x,\mathbf{v}), t_{out}(x,\mathbf{v})]$ where

$$t_{in}(x,\mathbf{v}) := \inf\{\tau \leqslant 0 : \mathcal{X}(s,x,\mathbf{v}) \in (0,1), \forall s \in (\tau,0)\},$$

$$t_{out}(x,\mathbf{v}) := \sup\{\tau \geqslant 0, \mathcal{X}(s,x,\mathbf{v}) \in (0,1), \forall s \in (0,\tau)\}.$$

We have three constant of motions for the system (C_e). Indeed, let us introduce the invariant functions defined for all $x \in [0,1]$ and $\mathbf{v} \in \mathbb{R}^3$ by

$$\begin{cases} E_1(x,\mathbf{v}) := \frac{1}{2}\left(v_x^2+v_y^2+v_z^2\right) - \frac{1}{\mu}\phi(x), & (4.34) \\ E_2(x,\mathbf{v}) := v_y, & (4.35) \\ E_3(x,\mathbf{v}) := v_z + \frac{\omega_i}{\mu}x. & (4.36) \end{cases}$$

Lemma 4.3.13. *The functions E_1,E_2 and E_3 are constant along the characteristics.*

Proof. For all $k \in \{1,2,3\}$, the function $c_k : t \in [t_{in}(x,\mathbf{v}), t_{out}(x,\mathbf{v})] \mapsto E_k(\mathcal{X}(t;x,\mathbf{v}),\mathcal{V}(t;x,\mathbf{v}))$ is differentiable on $(t_{in}(x,\mathbf{v}), t_{out}(x,\mathbf{v}))$ with $c_k' = 0$ thus c_k is constant. □

Thanks to Lemma 4.3.13, each characteristic curve is included a set $\mathcal{C}_{E_1,E_2,E_3}$ defined as the points $(x,y,v_x,v_y,v_z) \in [0,1]\times\mathbb{R}^3$ satisfying the system

$$\begin{cases} E_1 = \frac{1}{2}\left(v_x^2+v_y^2+v_z^2\right) - \frac{1}{\mu}\phi(x), \\ E_2 = v_y, \\ E_3 = v_z + \frac{\omega_i}{\mu}x. \end{cases}$$

where $(E_1,E_2,E_3) \in \mathbb{R}^3$ are now parameters. To study the set $\mathcal{C}_{E_1,E_2,E_3}$, it is convenient to define following function:

$$\forall E_3 \in \mathbb{R}, \forall x \in [0,1] \quad U_e(x,E_3) := \frac{(E_3 - \frac{\omega_i}{\mu}x)^2}{2} - \frac{1}{\mu}\phi(x). \tag{4.37}$$

Then we have for all $(E_1,E_2,E_3) \in \mathbb{R}^3$,$(x,v_x,v_y,v_z) \in \mathcal{C}_{E_1,E_2,E_3}$ if and only if:

$$\begin{cases} \frac{v_x^2}{2} = \mathcal{E} - U_e(x,E_3), \\ v_y = E_2, \\ v_z = E_3 + \omega_i x, \\ \mathcal{E} = E_1 - \frac{E_2^2}{2}. \end{cases}$$

The set $\mathcal{C}_{E_1,E_2,E_3}$ is not empty if and only if there is $x \in [0,1]$ such that $U_E(x,E_3) \leqslant \mathcal{E}$. When it is non-empty the set $\mathcal{C}_{E_1,E_2,E_3}$ consists of one or several curves in the four dimensional phase space $[0,1] \times \mathbb{R}^3$. In order to have a better representation of these curves, we introduce for each $E_3 \in \mathbb{R}$ the maximum of the function $U_e(.,E_3)$ namely:

$$\forall E_3 \in \mathbb{R}, \quad U_e^M(E_3) := \max_{x\in[0,1]} U_e(x,E_3).$$

It is well defined since the function $U_e(.,E_3)$ is continuous on $[0,1]$. We shall also need to split the physical space according to the locations of maxima in the interval [0,1]. We therefore introduce

$$x_M^-(E_3) := \min\{x \in [0,1] \mid U_e(x,E_3) = U_e^M(E_3)\}$$

as well as the two following sets defined for $E_3 \in \mathbb{R}$ by

$$D_{E_3} := \left\{(x,v_x) \in [0,1] \times \mathbb{R} \ \text{ s.t } |v_x| \geqslant \sqrt{2(U_e^M(E_3) - U_e(x,E_3))}\right\},$$

$$S_{E_3} := \left\{(x,v_x) \in [0,1] \times \mathbb{R} \text{ s.t } |v_x| = \sqrt{2(U_e^M(E_3) - U_e(x,E_3))}\right\}.$$

One has the decomposition $[0,1] \times \mathbb{R}^2 = D_{E_3} \cup D_{E_3}^c$ for all $E_3 \in \mathbb{R}$. The set S_{E_3} is called the separatrix curve of the (x,v_x) plane, it splits the (x,v_x) plane into two zones, namely:

a) these are the points of D_{E_3}: that are on a curve $\mathcal{C}_{E_1,E_2,E_3}$ that crosses the two boundaries $x=0$ and $x=1$.

b) these are the points that of $D_{E_3}^c$: that are on a curve $\mathcal{C}_{E_1,E_2,E_3}$ that do not cross the two boundaries $x=0$ and $x=1$, i.e, either they cross only one boundary or they never intersect the boundaries.

We decompose the four dimensional phase space as follows,

$$[0,1] \times \mathbb{R}^3 = \bigcup_{(E_2,E_3)\in\mathbb{R}^2} \left\{(x,v_x,v_y,v_z) \in [0,1] \times \mathbb{R}^3 \text{ s.t } (x,v_x) \in D_{E_3} \cup D_{E_3}^c,\ v_y = E_2,\ v_z = E_3 - \frac{\omega_i}{\mu}x\right\}.$$

Remark 23. *The family of sets defined for each $(E_2,E_3) \in \mathbb{R}^2$ by*

$$Q_{E_2,E_3} := \left\{(x,v_x,v_y,v_z) \in [0,1] \times \mathbb{R}^3 \ s.t\ (x,v_x) \in D_{I_3} \cup D_{I_3}^c,\ v_y = E_2,\ v_z = E_3 - \frac{\omega_i}{\mu}x\right\}$$

is a partition of the phase space $[0,1]\times\mathbb{R}^3$*. Each of these sets is the intersection of the portion of two orthogonal hyperplanes, namely:*

$$\mathcal{P}_{E_2} \text{ is the hyperplane with normal vector } (0,0,1,0)^t \text{ passing through } (0,0,E_2,0)^t,$$

and

$$\mathcal{P}_{E_3} \text{ is the hyperplane with normal vector } (\frac{\omega_i}{\mu},0,0,1)^t \text{ passing through } (0,0,0,E_3)^t.$$

Remark 24. *With this partitioning of the phase space, we know whether a point* $(x,v_x,v_y,v_z)\in[0,1]\times\mathbb{R}^3$ *is originally on a characteristic curve that reaches the boundaries or not. Indeed, it suffices to check if* (x,v_x) *belongs to* D_{E_3} *where* $E_3=v_z+\frac{\omega_i}{\mu}x$*.*

Remark 25. *It is possible that there are closed characteristic curves that do not cross the boundaries. If so, the solution to the linear Vlasov equation is not unique because on these curves the solution is not determined by the incoming boundary conditions, it can take arbitrary values. In this work, we shall consider that the solutions must vanish on closed characteristic curves.*

We do not give an illustration of the phase space with the potential U_e since the partitioning of the phase space follows the same idea as for the ions. We refer the reader to the Figure 4.2. We nevertheless stress the fact that the main difference between the ion potential U_i and the electron potential U_e is that U_e is convex while U_i is not necessarily. To finish with this section, let us introduce the following objects that are meant to facilitate the reading of the following section dedicated to the construction of a solution. We define for all $x\in[0,1]$ and $E_3\in\mathbb{R}$,

$$\delta_e^0(x,I_3):=U_e(0,E_3)-U_e(x,E_3),\quad \delta_e^1(x,E_3):=U_e(1,E_3)-U_e(x,E_3),\quad \delta_e^M(x,E_3):=U_e^M(E_3)-U_e(x,E_3)\geqslant 0,$$

$$V^0(x,E_3):=\{v_x\in\mathbb{R}\,|v_x^2\geqslant 2\delta_e^0(x,E_3)\},$$

Construction of a solution for the electrons

Because of the boundary conditions and the geometry of the characteristics, we shall consider weak solutions for the Vlasov equation.

Definition 4.3.14 (Weak solution)**.** *Let* $v_x f_e^{in}\in L^1_{loc}([0,+\infty)\times\mathbb{R}^2)$*. We say that* $f_e\in L^1_{loc}([0,1]\times\mathbb{R}^3)$ *is a weak solution of the Vlasov equation (4.14) iff for all* $\varphi\in C^1_c([0,1]\times\mathbb{R}^3)$ *with* $\varphi(1,\mathbf{v})=-\alpha\varphi(1,-v_x,v_y,v_z)$ *for* $v_x\geqslant 0$ *and* $\varphi(0,\mathbf{v})=0$ *for* $v_x\leqslant 0$ *we have*

$$-\int_0^1\int_{\mathbb{R}^3} f_e(x,\mathbf{v})\psi(x,\mathbf{v})dxd\mathbf{v}=\int_{\mathbb{R}^2}\int_{\mathbb{R}+} f_e^{in}(\mathbf{v})\varphi(0,\mathbf{v})v_xdv_xdv_ydv_z,$$

where $\psi(x,\mathbf{v}):=v_x\partial_x\varphi(x,\mathbf{v})+-\frac{1}{\mu}\left(-\partial_x\phi(x)\mathbf{e_x}+\omega_i\mathbf{v}\times\mathbf{b}\right).\nabla_v\varphi(x,\mathbf{v})$*.*

Using the fact that solutions of the linear Vlasov equations are constant along the characteristics, we are going to define a solution. We define the function f_e as follows: For all $(x,v_x,v_y,v_z)\in[0,1]\times\mathbb{R}^3$, let $E_2=v_y$ and $E_3=v_z+\frac{\omega_i}{\mu}x$ then we define

$$f_e(x,v_x,v_y,v_z)=\begin{cases}\text{ If } (x,v_x)\in D_{E_3}\setminus S_{E_3} \text{ then}\\ n_0f_e^{in}(\sqrt{v_x^2-2\delta_e^0(x,E_3)},E_2,E_3) \text{ for } v_x>0,\\ \alpha n_0f_e^{in}(\sqrt{v_x^2-2\delta_e^0(x,E_3)},E_2,E_3) \text{ for } v_x<0,\\ \text{ If } (x,v_x)\in D_{E_3}^c \text{ then}\\ n_0f_e^{in}(\sqrt{v_x^2-2\delta_e^0(x,E_3)},E_2,E_3) \text{ if } (x,v_x)\in[0,x_M^-(E_3)]\times V^0(x,E_3),\\ 0 \text{ if } (x,v_x)\notin[0,x_M^-(E_3)]\times V^0(x,E_3).\end{cases}\tag{4.38}$$

Remark 26. *The set* $\bigcup_{(E_2,E_3)\in\mathbb{R}^2} \{(x,v_x,v_y,v_z)\in[0,1]\times\mathbb{R}^3$ *s.t* $(x,v_x)\in S_{E_3}, v_y=E_2, v_z=E_3-\frac{\omega_i}{\mu}x\}$ *being of measure zero (because it is a three dimensional volume) we have not defined* f_e *on this set. This is not necessary since we consider weak solutions that are functions almost everywhere defined.*

One can prove the following.

Proposition 4.3.15. *The function* f_e *defined by* (4.38) *is a weak solution of the Vlasov equation.*

First three moments of the electron distribution

Definition 4.3.16. *We define the density, the current density and the kinetic energy associated to the electron distribution* f_e *defined in* (4.38) *by the functions defined respectively for all* $x\in[0,1]$ *by*

$$n_e(x) := \int_{\mathbb{R}} f_e(x,\mathbf{v})d\mathbf{v}, \quad \gamma_e(x) := \int_{\mathbb{R}} f_e(x,\mathbf{v})v_x dv, \quad \mathcal{E}_e(x) := \frac{1}{2}\int_{\mathbb{R}} f_e(x,\mathbf{v})v_x^2 dv.$$

Using the definition of f_e and straightforward integration yields the following proposition.

Proposition 4.3.17. *(Electron density) For all* $x\in[0,1]$ *we have*

$$\begin{aligned} n_e(x) &= (1+\alpha)n_0 \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\sqrt{2\delta_e^M(x,E_3)}}^{+\infty} f_e^{in}(\sqrt{v_x^2-2\delta_e^0(x,E_3)},E_2,E_3)dv_x dI_3 dI_2 \\ &+ n_0\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\{|v_x|<\sqrt{2\delta_e^M(x,E_3)}\}\cap\{v_x^2\geqslant 2\delta_e^0(x,E_3)\}} \mathbf{1}_{[0,x_M^-(E_3)]}(x) f_e^{in}(\sqrt{v_x^2-2\delta_e^0(x,E_3)},E_2,E_3)dv_x dE_3 dE_2. \end{aligned}$$

We can also compute the current and the kinetic energy.

Proposition 4.3.18. *(Electron current density) For all* $x\in[0,1]$

$$\gamma_e(x) = (1-\alpha)n_0\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\sqrt{2\delta_e^M(0,E_3)}}^{+\infty} f_e^{in}(w_x,E_2,E_3)w_x dw_x dE_2 dE_3.$$

Proof. The proof follows from an integration and cancellations due to symetries. □

Proposition 4.3.19. *(Electron kinetic energy) For all* $x\in[0,1]$

$$\begin{aligned} 2\mathcal{E}_e(x) &= (1+\alpha)n_0 \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\sqrt{2\delta_e^M(x,E_3)}}^{+\infty} f_e^{in}(\sqrt{v_x^2-2\delta_e^0(x,E_3)},E_2,E_3)v_x^2 dv_x dE_3 dE_2 \\ &+ n_0\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\{|v_x|<\sqrt{2\delta_e^M(x,E_3)}\}\cap\{v_x^2\geqslant 2\delta_e^0(x,E_3)\}} \mathbf{1}_{[0,x_M^-(E_3)]}(x) f_e^{in}(\sqrt{v_x^2-2\delta_e^0(x,E_3)},E_2,E_3)v_x^2 dv_x dE_3 dE_2. \end{aligned}$$

Let us now clarify why the assumptions 4.23 will be useful:

- We see the dependence on ϕ of the expressions for the density, the current and the kinetic energy through the number $x_M^-(E_3)$, $U_e^M(E_3)$ and the function δ_e^0.
- The numbers $x_M^-(E_3)$, $U_e^M(E_3)$ are not explicit. At the linear level, this does not look like a difficulty. Nevertheless at the non linear level and especially when we want to determine the wall potential ϕ_w and the density n_0, it appears to be convenient to know these numbers independently of ϕ.

Lemma 4.3.20. *Let $\underline{E}$ the function defined for all $x \in (0,1]$ by*

$$\underline{E}(x) := \frac{1}{\omega_i x}\left(\frac{\omega_i^2 x^2}{2\mu} - \phi(x)\right), \text{ and the number } E(\phi_w) := \frac{1}{\omega_i}\left(\frac{\omega_i^2}{2\mu} - \phi_w\right).$$

Assume (4.23), then one has:

a) For all $E_3 \in \mathbb{R}$, the function $U_e(.,E_3)$ is convex.

b) $U_e^M(E_3) = U_e(0,E_3) \Leftrightarrow E_3 \geqslant E(\phi_w)$ and $x_M^-(E_3) = 0$ for all $E_3 \geqslant E(\phi_w)$.

c) $U_e^M(E_3) = U_e(1,E_3) \Leftrightarrow E_3 \leqslant E(\phi_w)$ and $x_M^-(E_3) = 1$ for all $E_3 < E(\phi_w)$.

d) $\forall E_3 \in \mathbb{R}$, $\forall x \in (0,1]$, $U_e(x,E_3) \geqslant U_e(0,E_3) \Leftrightarrow E_3 \leqslant \underline{E}(x)$

Proof. a) Let $E_3 \in \mathbb{R}$. For a.e every $x \in [0,1]$ we have

$$U_e''(x,E_3) = \frac{\omega_i^2}{\mu^2} - \frac{1}{\mu}\phi''(x) = \frac{1}{\mu}\left(\frac{\omega_i^2}{\mu} - \phi''(x)\right).$$

Since $\phi''(x) \leqslant \frac{\omega_i^2}{\mu}$ we deduce that $U_e''(.,E_3) \geqslant 0$ almost everywhere in $[0,1]$ and thus $U_e(.,E_3)$ is convex.

b),c) From a) we deduce that for all $E_3 \in \mathbb{R}$, $U_e^M(E_3) = \max\{U_e(0,E_3), U_e(1,E_3)\}$, and

$$U_e(0,E_3) \leqslant U_e(1,E_3) \text{ iff } E_3 \leqslant E(\phi_w).$$

Thus $U_e^M(E_3) = U_e(0,E_3)$ iff $E_3 \geqslant E(\phi_w)$ and $U_e^M(E_3) = U_e(1,E_3)$ iff $E_3 \leqslant E(\phi_w)$.

d) It follows from a simple computation.

□

Remark 27. *This lemma has also an interpretation in terms of physical situations we consider. As compared with the ions, for the electrons there are several scenarios, let us describe two of them: either a particle enters at $x = 0$ with a positive v_x velocity and with a velocity $v_z \geqslant E(\phi_w)$ and then the particle is fated to reach $x = 1$. Or it enters at $x = 0$ with a positive v_x velocity and with a velocity $v_z < E(\phi_w)$ and then the particle has two possible destinies, either it has a sufficiently large v_x velocity and then it reaches $x = 1$ (the wall) or it returns at $x = 0$ with a negative velocity because its v_x velocity was insufficient. It is quite interesting to remark that both the limits $\omega_i \to 0^+$ and $\omega_i \to +\infty$ leads to our last scenario since $E(\phi_w) \to +\infty$. That is, low energetic particles that tend be naturally confined by either the magnetic field or the electrostatic potential, and high energetic particles that reach the wall anyway.*

Remark 28. *Because of the convexity of U_e, it is easy to see that for all $x \in [0,1]$, $\delta_e^M(x,E_3) = \delta_e^0(x,E_3)$ if $E_3 \geqslant E(\phi_w)$ and $\delta_e^M(x,E_3) = \delta_e^1(x,E_3)$ if $E_3 \leqslant E(\phi_w)$. Moreover, the set $V^0(x,E_3)$ is such that*

$$\text{if } x \in (0,1] \quad V^0(x,E_3) = \mathbb{R} \text{ iff } E_3 \leqslant \underline{E}(x),$$

$$\text{if } x = 0, \quad V^0(x,E_3) = \mathbb{R}\ \forall E_3 \in \mathbb{R}.$$

The lemma 4.3.20 enables to give another expression of the density, the current and the kinetic energy where firstly, the numbers $x_M^-(E_3)$ and $U_e^M(E_3)$ no longer depend on ϕ and secondly, it shows off the dependence with respect to the wall potential ϕ_w.

Proposition 4.3.21. *Assume* (4.22)-(4.24) *then for all* $x \in [0,1]$ *one has*

$$
\begin{aligned}
n_e(x) =& n_0(1+\alpha)\int_{\mathbb{R}}\int_{-\infty}^{E(\phi_w)}\int_{\sqrt{2\delta_e^1(0,E_3)}}^{+\infty}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_xdE_3dE_2\\
& n_0(1+\alpha)\int_{\mathbb{R}}\int_{E(\phi_w)}^{+\infty}\int_{0}^{+\infty}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_xdE_3dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)\geqslant E(\phi_w)}(x)\int_{\mathbb{R}}\int_{-\infty}^{E(\phi_w)}\int_{\sqrt{-2\delta_e^0(x,E_3)}}^{\sqrt{2\delta_e^1(0,E_3)}}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_xdE_3dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)< E(\phi_w)}(x)\int_{\mathbb{R}}\int_{-\infty}^{\underline{E}(x)}\int_{\sqrt{-2\delta_e^0(x,E_3)}}^{\sqrt{2\delta_e^1(0,E_3)}}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_xdE_3dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)< E(\phi_w)}(x)\int_{\mathbb{R}}\int_{\underline{E}(x)}^{E(\phi_w)}\int_{0}^{\sqrt{2\delta_e^1(0,E_3)}}\frac{f_e^{in}(w_x,E_2,E_3)w_x}{\sqrt{w_x^2+2\delta_e^0(x,E_3)}}dw_xdE_3dE_2
\end{aligned}
$$

where the function $\underline{E}$ *is extended by contuinity at* $x=0$ *with* $\underline{E}(0) = -\frac{\phi'(0)}{\omega_i}$.

$$
\begin{aligned}
\gamma_e(x) =& (1-\alpha)n_0\int_{\mathbb{R}}\int_{-\infty}^{E(\phi_w)}\int_{\sqrt{2\delta_e^1(0,E_3)}}^{+\infty} f_e^{in}(w_x,E_2,E_3)w_xdw_xdE_3dE_2\\
& + (1-\alpha)n_0\int_{\mathbb{R}}\int_{E(\phi_w)}^{+\infty}\int_{0}^{+\infty} f_e^{in}(w_x,E_2,E_3)w_xdw_xdE_3dE_2.
\end{aligned}
$$

$$
\begin{aligned}
2\mathcal{E}_e(x) =& n_0(1+\alpha)\int_{\mathbb{R}}\int_{-\infty}^{E(\phi_w)}\int_{\sqrt{2\delta_e^1(0,E_3)}}^{+\infty} f_e^{in}(w_x,E_2,E_3)w_x\sqrt{w_x^2+2\delta_e^0(x,E_3)}dw_xdE_3dE_2\\
& n_0(1+\alpha)\int_{\mathbb{R}}\int_{E(\phi_w)}^{+\infty}\int_{0}^{+\infty} f_e^{in}(w_x,E_2,E_3)w_x\sqrt{w_x^2+2\delta_e^0(x,E_3)}dw_xdE_3dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)\geqslant E(\phi_w)}(x)\int_{\mathbb{R}}\int_{-\infty}^{E(\phi_w)}\int_{\sqrt{-2\delta_e^0(x,E_3)}}^{\sqrt{2\delta_e^1(0,E_3)}} f_e^{in}(w_x,E_2,E_3)w_x\sqrt{w_x^2+2\delta_e^0(x,E_3)}dw_xdE_3dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)< E(\phi_w)}(x)\int_{\mathbb{R}}\int_{-\infty}^{\underline{E}(x)}\int_{\sqrt{-2\delta_e^0(x,E_3)}}^{\sqrt{2\delta_e^1(0,E_3)}} f_e^{in}(w_x,E_2,E_3)w_x\sqrt{w_x^2+2\delta_e^0(x,E_3)}dw_xdE_3dE_2\\
& + 2n_0\mathbf{1}_{\underline{E}(x)< E(\phi_w)}(x)\int_{\mathbb{R}}\int_{\underline{E}(x)}^{E(\phi_w)}\int_{0}^{\sqrt{2\delta_e^1(0,E_3)}} f_e^{in}(w_x,E_2,E_3)w_x\sqrt{w_x^2+2\delta_e^0(x,E_3)}dw_xdE_3dE_2
\end{aligned}
$$

4.3.2 Maxwellian incoming electron boundary condition

Since we are interested to describe the transition between the plasma and the wall, we shall consider a Maxwellian boundary conditions for the electrons. Indeed, as mentionned in [63] electrons in the core of the plasma are well described by a full Maxwellian distribution as a matter of fact, the boundary conditions is taken of the form

$$
F_e^{in}(V_x,V_y,V_z) := 2\left(\frac{m_e}{2\pi k_B T_e}\right)^{\frac{3}{2}} e^{-\dfrac{m_e(V_x^2+V_y^2+V_z^2)}{2k_BT_e}} \qquad \forall \mathbf{V}\in(0,+\infty)\times\mathbb{R}^2. \tag{4.39}
$$

It gives in terms of dimensionless variables

$$f_e^{in}(\mathbf{v}) := 2\left(\frac{\mu}{2\pi}\right)^{\frac{3}{2}} e^{-\dfrac{\mu(v_x^2+v_y^2+v_z^2)}{2}} \qquad \forall \mathbf{v} \in (0,+\infty)\times\mathbb{R}^2, \tag{4.40}$$

and note that $\displaystyle\int_{\mathbb{R}^2\times\mathbb{R}^+} f_e^{in}(\mathbf{v})d\mathbf{v} = 1$. The electron density is then given for all $x \in [0,1]$ by

$$\begin{aligned}
n_e(x) &= (1+\alpha)n_0\frac{e^{\phi(x)}}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\operatorname{erfc}\left(\sqrt{\mu\delta_e^1(x,E_3)}\right)d\tilde{E}_3\\
&+(1+\alpha)n_0\frac{e^{\phi(x)}}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2}\operatorname{erfc}\left(\sqrt{\mu\delta_e^0(x,E_3)}\right)d\tilde{E}_3\\
&+2n_0\mathbf{1}_{\underline{E}(x)\geqslant E(\phi_w)}(x)\frac{e^{\phi(x)}}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\operatorname{erf}\left(\sqrt{\mu\delta_e^1(x,E_3)}\right)d\tilde{E}_3\\
&+2n_0\mathbf{1}_{\underline{E}(x)< E(\phi_w)}(x)\frac{e^{\phi(x)}}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(\underline{E}(x)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\operatorname{erf}\left(\sqrt{\mu\delta_e^1(x,E_3)}\right)d\tilde{E}_3\\
&+2n_0\mathbf{1}_{\underline{E}(x)< E(\phi_w)}(x)\frac{e^{\phi(x)}}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(\underline{E}(x)-\frac{\omega_i x}{\mu})}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\left(\operatorname{erf}\left(\sqrt{\mu\delta_e^1(x,E_3)}\right)-\operatorname{erf}\left(\sqrt{\mu\delta_e^0(x,E_3)}\right)\right)d\tilde{E}_3.
\end{aligned} \tag{4.41}$$

where $E_3 = \sqrt{\frac{2}{\mu}}\tilde{E}_3 + \frac{\omega_i x}{\mu}$, erf is the error function defined for all $x \in \mathbb{R}$ by

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

and erfc $= 1 -$ erf is the complementary error function. Since ϕ is non positive and $|\operatorname{erf}(x)| \leqslant 1$, $|\operatorname{erfc}(x)| \leqslant 2$ for all $x \in \mathbb{R}$, one has the bound for all $x \in [0,1]$,

$$|n_e(x)| \leqslant 2\sqrt{2}n_0(2\alpha+8).$$

The current density is constant in space and given for all $x \in [0,1]$ by

$$\begin{aligned}
\gamma_e(x) &= (1-\alpha)n_0\frac{e^{\phi_w}}{\sqrt{2\pi\mu}}\left(1+\operatorname{erf}\left(\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i}{\mu})\right)\right)\\
&+(1-\alpha)n_0\frac{\operatorname{erfc}\left(\sqrt{\frac{\mu}{2}}E(\phi_w)\right)}{\sqrt{2\pi\mu}}.
\end{aligned} \tag{4.42}$$

4.4 Determination of the wall (floating) potential ϕ_w and the electron reference density n_0

We are going to show that there exists $(n_0, \phi_w) \in (0, +\infty) \times \mathbb{R}^-$ depending only on f_i^{in},α,ω_i and μ such that the system

$$\begin{cases} \gamma_i(x) = \gamma_e(x) \quad \forall x \in [0,1], \\ n_i(0) = n_e(0), \end{cases} \tag{4.43}$$

where γ_i is given (4.32), γ_e by (4.42), n_i by (4.31) and n_e by (4.41) holds.

4.4.1 The Ampère equation

Generally speaking, the potential at the wall cannot be a priori specified as a physical parameter, because its value is determined from the ambipolarity of the flows. Therefore it is important to understand how it is determined in this model from other physical parameters. As mentioned in the introduction, the wall potential adjusts itself so that equal numbers of ions and electrons reach the wall per second. Following the idea in [63] Section 2.6 page 79 we have that the ambipolarity reads for all $\alpha \in [0, 1]$

$$\begin{aligned} &\int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) w_x dw_x dI_2 dI_3 = (1-\alpha) n_0 \frac{e^{\phi_w}}{\sqrt{2\pi\mu}} \left(1 + \operatorname{erf}\left(\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i}{\mu})\right)\right) \\ &+ (1-\alpha) n_0 \frac{\operatorname{erfc}\left(\sqrt{\frac{\mu}{2}} E(\phi_w)\right)}{\sqrt{2\pi\mu}} \end{aligned} \tag{4.44}$$

Remark 29. *We see that for $\alpha = 1$, the above equation implies*

$$\int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) w_x dw_x dI_2 dI_3 = 0$$

and because f_i^{in} is non negative thus $f_i^{in}(\mathbf{v}) = 0$ for a.e $\mathbf{v} \in (0, +\infty) \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2})$. The model is therefore of no physical interest and it shows that the absorbing properties of the wall play a significant role in the possibility to determine the floating potential.

4.4.2 The neutrality relation

We now express the neutrality relation with n_i given by (4.31) and n_e given by (4.41). One has

$$n_i(0) = \int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) dw_x dI_3 dI_2,$$

$$\begin{aligned} n_e(0) = &\frac{(1+\alpha) n_0}{\sqrt{\pi}} \left(\int_{-\infty}^{\sqrt{\frac{\mu}{2}} E(\phi_w)} e^{-\tilde{E_3}^2} \operatorname{erfc}\left(\sqrt{\mu \delta_e^1(0, E_3)}\right) d\tilde{E}_3 + \frac{\sqrt{\pi}}{2} \operatorname{erfc}\left(\frac{\sqrt{\mu}}{2} E(\phi_w)\right)\right) \\ &+ \frac{2 n_0}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}} E(\phi_w)} e^{-\tilde{E_3}^2} \operatorname{erf}\left(\sqrt{\mu \delta_e^1(0, E_3)}\right) d\tilde{E}_3 \text{ where } E_3 = \sqrt{\frac{2}{\mu}} \tilde{E}_3. \end{aligned}$$

Using the fact that for all $x \in \mathbb{R}$, $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ for all $x \in \mathbb{R}$ yields simplification in the expression of $n_e(0)$. Indeed we have

$$\int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\phi_w)} e^{-\tilde{E_3}^2} \mathrm{erfc}\left(\sqrt{\mu\delta_e^1(0,E_3)}\right) d\tilde{E_3} = \frac{\sqrt{\pi}}{2}\left(2 - \mathrm{erfc}(\sqrt{\frac{\mu}{2}}E(\phi_w))\right)$$
$$- \int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\phi_w)} e^{-\tilde{E_3}^2} \mathrm{erf}(\sqrt{\mu\delta_e^1(0,E_3)}) d\tilde{E_3}$$

and thus we obtain

$$n_e(0) = (1+\alpha)n_0 + \frac{(1-\alpha)n_0}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\phi_w)} e^{-\tilde{E_3}^2} \mathrm{erf}(\sqrt{\mu\delta_e^1(0,E_3)}) d\tilde{E_3}.$$

The neutrality relation therefore reads for all $\alpha \in [0,1]$,

$$\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x,I_2,I_3)dw_x dI_3 dI_2 = (1+\alpha)n_0 + \frac{(1-\alpha)n_0}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\phi_w)} e^{-\tilde{E_3}^2} \mathrm{erf}(\sqrt{\mu\delta_e^1(0,E_3)}) d\tilde{E_3}.$$

Note that n_0 can be expressed as

$$n_0 = \frac{\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x,I_2,I_3)dw_x dI_3 dI_2}{(1+\alpha) + \frac{(1-\alpha)}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\phi_w)} e^{-\tilde{E_3}^2} \mathrm{erf}(\sqrt{\mu\delta_e^1(0,E_3)}) d\tilde{E_3}}$$

Remark that since $\phi(1) = \phi_w$, we have for all E_3 $\delta_e^1(0,E_3) = -\frac{\phi_w}{\mu} + \frac{\omega_i^2}{2\mu^2} - \frac{E_3\omega_i}{\mu}$. Let us then introduce the function

$$h : (E_3,\phi_w) \in \mathbb{R}\times(-\infty,0] \mapsto -\frac{\phi_w}{\mu} + \frac{\omega_i^2}{2\mu^2} - \frac{E_3\omega_i}{\mu}. \tag{4.45}$$

Therefore n_0 is given as a function of ϕ_w by

$$n_0 = \frac{\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x,I_2,I_3)dw_x dI_3 dI_2}{(1+\alpha) + \frac{(1-\alpha)}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\phi_w)} e^{-\tilde{E_3}^2} \mathrm{erf}(\sqrt{\mu h(E_3,\phi_w)}) d\tilde{E_3}} \tag{4.46}$$

4.4.3 The non linear equation on the floating potential

Let us define for convenience

$$M_0(f_i^{in},\omega_i) := \int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x,I_2,I_3)dw_x dI_3 dI_2,$$

$$M_1(f_i^{in},\omega_i) := \int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x,I_2,I_3)w_x dw_x dI_3 dI_2.$$

Substituting the expression of n_0 given by (4.46) in the ambipolarity equality (4.44) yields the non linear relation

$$\mathcal{W}(\phi_w) = b \tag{4.47}$$

where the function $\mathcal{W} : \mathbb{R}^- \to \mathbb{R}$ is defined for all $\psi \leqslant 0$ by

$$\begin{aligned}\mathcal{W}(\psi) = &\frac{(1-\alpha)M_0(f_i^{in},\omega_i)}{\sqrt{2\pi\mu}}\left[e^{\psi}\left(1+\mathrm{erf}\left(\sqrt{\frac{\mu}{2}}(E(\psi)-\frac{\omega_i}{\mu})\right)+\mathrm{erfc}(\sqrt{\frac{\mu}{2}}E(\psi))\right)\right] \\ &-\frac{(1-\alpha)M_1(f_i^{in},\omega_i)}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\psi)} e^{-\tilde{E_3}^2}\,\mathrm{erf}(\sqrt{\mu h(E_3,\psi)})d\tilde{E_3},\end{aligned} \tag{4.48}$$

where $E(\psi) = \frac{1}{\omega_i}\left(\frac{\omega_i^2}{2\mu}-\psi\right)$, and $b = (1+\alpha)M_1(f_i^{in},\omega_i)$. The main result of this section is the following.

Theorem 4.4.1. *(Existence and uniqueness of the floating potential) Let $\alpha \in [0,1)$ and $\omega_i > 0$. Assume that $f_i^{in} \in L^1(\mathbb{R}^+ \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2}); \mathbb{R}^+)$ and such that $v_x f_i^{in} \in L^1(\mathbb{R}^+ \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2}))$. The equation (4.47) has a unique non positive solution if and only if*

$$0 < \frac{M_1(f_i^{in},\omega_i)}{M_0(f_i^{in},\omega_i)} \leqslant s_1(\alpha,\omega_i,\mu), \tag{4.49}$$

where

$$s_1(\alpha,\omega_i,\mu) := \frac{2(1-\alpha)\,\mathrm{erfc}(\frac{\omega_i}{2\sqrt{2\mu}})}{\sqrt{2\pi\mu}\left((1+\alpha)+\frac{(1-\alpha)}{\sqrt{\pi}}\int_{-\infty}^{\frac{\omega_i}{2\sqrt{2\mu}}} e^{-\tilde{E_3}^2}\,\mathrm{erf}(\sqrt{\mu h(E_3,0)})d\tilde{E_3}\right)}. \tag{4.50}$$

Proof. One has for all $\psi \leqslant 0$,

$$\mathcal{W}(\psi) = \mathcal{W}_1(\psi) + \mathcal{W}_2(\psi) + \mathcal{W}_3(\psi)$$

with

$$\begin{aligned}\mathcal{W}_1(\psi) &= \frac{(1-\alpha)M_0(f_i^{in},\omega_i)}{\sqrt{2\pi\mu}}e^{\psi}\left(1+\mathrm{erf}\left(\sqrt{\frac{\mu}{2}}(E(\psi)-\frac{\omega_i}{\mu})\right)\right),\\ \mathcal{W}_2(\psi) &= \frac{(1-\alpha)M_0(f_i^{in},\omega_i)}{\sqrt{2\pi\mu}}\,\mathrm{erfc}(\sqrt{\frac{\mu}{2}}E(\psi)),\\ \mathcal{W}_3(\psi) &= -\frac{(1-\alpha)M_1(f_i^{in},\omega_i)}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\psi)} e^{-\tilde{E_3}^2}\,\mathrm{erf}(\sqrt{\mu h(E_3,\psi)})d\tilde{E_3}.\end{aligned}$$

These functions are differentiable on $(-\infty, 0]$ with

$$\begin{aligned}\mathcal{W}_1'(\psi) &= \frac{(1-\alpha)M_0(f_i^{in},\omega_i)}{\sqrt{2\pi\mu}}e^{\psi}\left(1+\mathrm{erf}\left(\sqrt{\frac{\mu}{2}}(E(\psi)-\frac{\omega_i}{\mu})\right)\right)-\frac{(1-\alpha)M_0}{\omega_i\pi}e^{-\mu U_e(1,E(\psi))},\\ \mathcal{W}_2'(\psi) &= \frac{(1-\alpha)M_0(f_i^{in},\omega_i)}{\omega_i\pi}e^{-\mu U_e(0,E(\psi))},\\ \mathcal{W}_3'(\psi) &= \frac{(1-\alpha)M_1(f_i^{in},\omega_i)}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\psi)}\frac{e^{-\tilde{E_3}^2-\mu h(E_3,\psi)}}{2\sqrt{\mu h(E_3,\psi)}}d\tilde{E_3}.\end{aligned}$$

Note that the integrand in $\mathcal{W}_3'(\psi)$ as a singularity at $\tilde{E_3} = \sqrt{\frac{\mu}{2}}E(\psi)$ since $h(E(\psi),\psi) = 0$. This singularity is integrable and $\mathcal{W}_3'$ is well defined for all $\psi \leqslant 0$. Also note that $U_e(0,E(\psi)) = U_e(1,E(\psi))$ and for all

$\psi \leqslant 0$ so that

$$\begin{aligned}
&\mathcal{W}_1'(\psi) + \mathcal{W}_2'(\psi) + \mathcal{W}_3'(\psi) = \\
&\frac{(1-\alpha)M_0(f_i^{in},\omega_i)}{\sqrt{2\pi\mu}} e^{\psi} \left(1 + \operatorname{erf}\left(\sqrt{\frac{\mu}{2}}(E(\psi) - \frac{\omega_i}{\mu})\right)\right) \\
&+ \frac{(1-\alpha)M_1(f_i^{in},\omega_i)}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}E(\psi)} \frac{e^{-\tilde{E_3}^2 - \mu h(E_3,\psi)}}{2\sqrt{\mu h(E_3,\psi)}} d\tilde{E}_3.
\end{aligned}$$

Note that $\mathcal{W}'$ is positive on $(-\infty, 0]$ and therefore the function $\mathcal{W}$ is increasing on $(-\infty, 0]$. Moreover it is continuous with $\lim\limits_{\psi \to -\infty} \mathcal{W}(\psi) = -(1-\alpha)M_1(f_i^{in})$, therefore the equation (4.47) has a unique a non positive solution if and only if $-(1-\alpha)M_1(f_i^{in},\omega_i) < (1+\alpha)M_1(f_i^{in},\omega_i) \leqslant \mathcal{W}(0)$. Besides, ne has

$$\begin{aligned}
\mathcal{W}(0) = &\frac{(1-\alpha)M_0(f_i^{in},\omega_i)}{\sqrt{2\pi\mu}} \left(1 + \operatorname{erf}(-\frac{\omega_i}{2\sqrt{2\mu}}) + \operatorname{erfc}(\frac{\omega_i}{2\sqrt{2\mu}})\right) \\
&- \frac{(1-\alpha)M_1(f_i^{in},\omega_i)}{\sqrt{\pi}} \int_{-\infty}^{\frac{\omega_i}{2\sqrt{2\mu}}} e^{-\tilde{E_3}^2} \operatorname{erf}(\sqrt{\mu h(E_3,0)}) d\tilde{E}_3.
\end{aligned}$$

Then it suffices to remark that $\operatorname{erf}(-\frac{\omega_i}{2\sqrt{2\mu}}) = -\operatorname{erf}(\frac{\omega_i}{2\sqrt{2\mu}}) = \operatorname{erfc}(\frac{\omega_i}{2\sqrt{2\mu}}) - 1$ to get that

$$\begin{aligned}
\mathcal{W}(0) = &\frac{2(1-\alpha)M_0(f_i^{in},\omega_i)}{\sqrt{2\pi\mu}} \\
&- \frac{(1-\alpha)M_1(f_i^{in},\omega_i)}{\sqrt{\pi}} \int_{-\infty}^{\frac{\omega_i}{2\sqrt{2\mu}}} e^{-\tilde{E_3}^2} \operatorname{erf}(\sqrt{\mu h(E_3,0)}) d\tilde{E}_3.
\end{aligned}$$

It then easy to conclude. □

Remark 30. *In the limit $\omega_i \to 0^+$ we have that $s_1(\alpha,\omega_i,\mu) \to \frac{(1-\alpha)}{(1+\alpha)}\sqrt{\frac{2}{\pi\mu}}$. This is exactly the upper bound we found in the electrostatic case see (2.40).*

Remark 31. *This theorem states the existence of wall potential $\phi_w \leqslant 0$ solution to (4.47) and by the way the existence of the pair $(\phi_w, n_0) \in \mathbb{R}^- \times (0,+\infty)$ where n_0 is given by (4.46) solution to the system (4.43). However, we guess that we cannot take ω_i too large. Indeed the upperbound in (4.49) limits the ion flux perpendicular to the wall. In the limit $\omega_i \to +\infty$ one has $s_1(\alpha,\omega_i,\mu) \to 0$ and similarly $\frac{M_1(f_i^{in},\omega_i)}{M_0(f_i^{in},\omega_i)} \to 0$. However it could be possible that for a given ion boundary condition f_i^{in} there is some critical value $\omega_i^c > 0$ such that for all $\omega_i > \omega_i^c$, $\frac{M_1(f_i^{in},\omega_i)}{M_0(f_i^{in},\omega_i)} > s_1(\alpha,\omega_i,\mu)$. This is what we observe at a numerical level.*

4.5 Mathematical study of the non linear Poisson problem

We are going to introduce the functional framework to study the problem (NLP-MAG) in the case of an incoming Maxwellian electron distribution. We therefore consider that $\alpha \in [0,1)$, $\omega_i > 0$, $\mu > 0$ and $f_i^{in} \in (L^1 \cap L^\infty)(\mathbb{R}^+ \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2}))$ satisfying (4.49) are given parameters as well as the normalized Debye length ε. The wall potential $\phi_w \leqslant 0$ will then be the solution of (4.47) and the electron boundary condition will be of the form (4.40) where the reference density n_0 is defined by (4.46). In particular, the (NLP-MAG) problem reformulates as follows : Let $\alpha \in [0,1)$, f_i^{in} satisfying (4.49) and $\varepsilon > 0$. Find $\phi : [0,1] \to \mathbb{R}$ solution of

$$\text{(NLP-MMAG)} \ : \begin{cases} -\varepsilon^2 \dfrac{d^2}{dx^2}\phi(x) = (n_i - n_e)(x) \text{ for all } x \in (0,1) \\ \text{with Dirichlet boundary conditions} \\ \phi(0) = 0 \text{ and } \phi(1) = \phi_w \end{cases} \tag{4.51}$$

where

$$n_i(x) = \int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} \frac{f_i^{in}(w_x, I_2, I_3) w_x}{\sqrt{w_x^2 + 2\delta_i^0(x, I_3)}} d_w dI_3 dI_2$$

and

$$\begin{aligned} n_e(x) &= (1+\alpha) n_0 \frac{e^{\phi(x)}}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erfc}\left(\sqrt{\mu \delta_e^1(x, E_3)}\right) d\tilde{E}_3 \\ &+ (1+\alpha) n_0 \frac{e^{\phi(x)}}{\sqrt{\pi}} \int_{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2} \operatorname{erfc}\left(\sqrt{\mu \delta_e^0(x, E_3)}\right) d\tilde{E}_3 \\ &+ 2n_0 \mathbf{1}_{\underline{E}(x) \geqslant E(\phi_w)}(x) \frac{e^{\phi(x)}}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erf}\left(\sqrt{\mu \delta_e^1(x, E_3)}\right) d\tilde{E}_3 \\ &+ 2n_0 \mathbf{1}_{\underline{E}(x) < E(\phi_w)}(x) \frac{e^{\phi(x)}}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(\underline{E}(x) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erf}\left(\sqrt{\mu \delta_e^1(x, E_3)}\right) d\tilde{E}_3 \\ &+ 2n_0 \mathbf{1}_{\underline{E}(x) < E(\phi_w)}(x) \frac{e^{\phi(x)}}{\sqrt{\pi}} \int_{\sqrt{\frac{\mu}{2}}(\underline{E}(x) - \frac{\omega_i x}{\mu})}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \left(\operatorname{erf}\left(\sqrt{\mu \delta_e^1(x, E_3)}\right) - \operatorname{erf}\left(\sqrt{\mu \delta_e^0(x, E_3)}\right)\right) d\tilde{E}_3. \end{aligned}$$

with $E_3 = \sqrt{\frac{2}{\mu}}\tilde{E}_3 + \frac{\omega_i x}{\mu}$, and where the function $\underline{E}$ defined for all $x \in (0,1]$ by $\underline{E}(x) = \frac{1}{\omega_i x}\left(\frac{\omega_i^2 x^2}{2\mu} - \phi(x)\right)$.

Remark 32. *Note that the difference of the two terms* $\left(\operatorname{erf}\left(\sqrt{\mu \delta_e^1(x, E_3)}\right) - \operatorname{erf}\left(\sqrt{\mu \delta_e^0(x, E_3)}\right)\right)$ *is non negative for all* $x \in [0,1]$ *and* $E_3 \leqslant E(\phi_w)$.

However sufficient it is, we must precise where the non linearity in the Poisson equation stands. To do so, we introduce the polynomials

$$\forall (x, I_3) \in [0,1] \times \mathbb{R}, \quad p_i^0(x, I_3) := -\frac{\omega_i^2 x^2}{2} - I_3 \omega_i x \tag{4.52}$$

$$\forall (x, E_3) \in [0,1] \times \mathbb{R}, \quad p_e^0(x, E_3) := -\frac{\omega_i^2 x^2}{2\mu} + E_3 \omega_i x \tag{4.53}$$

$$\forall (x, E_3) \in [0,1] \times \mathbb{R}, \quad p_e^1(x, E_3) := \frac{\omega_i^2}{2\mu}(1 - x^2) + E_3 \omega_i (x - 1) - \phi_w. \tag{4.54}$$

and the function $\mathcal{E}$ defined by

$$\mathcal{E}(x, t) := \frac{1}{\omega_i x}\left(\frac{\omega_i^2 x^2}{2\mu} - t\right) \text{ for } x \in (0,1] \text{ and } t \leqslant 0. \tag{4.55}$$

The densities n_i and n_e can therefore be re-written in the form

$$n_i(x) = \rho_i(x, \phi(x)), n_e(x) = \rho_e(x, \phi(x)) \quad \text{for all } x \in (0,1),$$

where the function ρ_i is defined for all $x \in [0,1]$ and $t \leqslant 0$ by

$$\rho_i(x,t) := \int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} \frac{f_i^{in}(w_x, I_2, I_3) w_x}{\sqrt{w_x^2 + 2(p_i^0(x, I_3) - t)}} dw_x dI_3 dI_2, \tag{4.56}$$

and the function ρ_e is defined for all $x \in (0,1]$ and $t \geqslant \frac{-\omega_i^2 x(1-x)}{2\mu} + x\phi_w$ by

$$\begin{aligned}
\rho_e(x,t) :=& (1+\alpha) n_0 \frac{e^t}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erfc}\left(\sqrt{p_e^1(x, E_3) + t}\right) d\tilde{E}_3 \\
&+ (1+\alpha) n_0 \frac{e^t}{\sqrt{\pi}} \int_{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2} \operatorname{erfc}\left(\sqrt{p_e^0(x, E_3) + t}\right) d\tilde{E}_3 \\
&+ 2n_0 \mathbf{1}_{\mathcal{E}(x,t) \geqslant E(\phi_w)}(x) \frac{e^t}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erf}\left(\sqrt{p_e^1(x, E_3) + t}\right) d\tilde{E}_3 \\
&+ 2n_0 \mathbf{1}_{\mathcal{E}(x,t) < E(\phi_w)}(x) \frac{e^t}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erf}\left(\sqrt{p_e^1(x, E_3) + t}\right) d\tilde{E}_3 \\
&+ 2n_0 \mathbf{1}_{\mathcal{E}(x,t) < E(\phi_w)}(x) \frac{e^t}{\sqrt{\pi}} \int_{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t) - \frac{\omega_i x}{\mu})}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \left(\operatorname{erf}\left(\sqrt{p_e^1(x, E_3) + t}\right) - \operatorname{erf}\left(\sqrt{p_e^0(x, E_3) + t}\right)\right) d\tilde{E}_3,
\end{aligned} \tag{4.57}$$

where $E_3 = \sqrt{\frac{2}{\mu}} \tilde{E}_3 + \frac{\omega_i x}{\mu}$.

Remark 33. *At this stage of the study, and compared to the study of the linear Vlasov system, we have not assumed anything about the profile of ϕ. We will only assume some pointwise bounds on ϕ that are necessary to ensure that square roots in the formula of ρ_i an ρ_e are well-defined.*

4.5.1 Regularity of ρ_i and ρ_e

Eventhough it is not sufficient to prove the existence of a solution to the non linear Poisson problem, it is nevertheless necessary to know the regularity of ρ_i and ρ_e.

Proposition 4.5.1. *The function ρ_i is well-defined and continuous on $[0,1] \times (-\infty, 0]$. Moreover, one has the bound*

$$|\rho_i(x,t)| \leqslant \|f_i^{in}\|_{L^1}$$

Proof. We begin with showing that ρ_i is well-defined and bounded. Notice that for all $(x,t) \in [0,1] \times (-\infty, 0]$ one has

$$\begin{aligned}
p_i^0(x, I_3) - t &= -\frac{\omega_i^2 x^2}{2} - I_3 \omega_i x - t \geqslant -\frac{\omega_i^2 x^2}{2} - I_3 \omega_i x \\
&\geqslant -\frac{\omega_i^2 x^2}{2} + \frac{\omega_i^2 x}{2} = \frac{\omega_i^2}{2} x(1-x) \geqslant 0 \quad \forall I_3 < -\frac{\omega_i}{2}.
\end{aligned}$$

Therefore for all $w_x > 0$, one has $\sqrt{w_x^2 + 2(p_i^0(x, i_3) - t)} \geqslant w_x$ and we deduce that for almost every $(w_x, I_2, I_3) \in (0, +\infty) \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2})$,

$$\frac{f_i^{in}(w_x, I_2, I_3) w_x}{\sqrt{w_x^2 + 2(p_i^0(x, I_3) - t)}} \leqslant f_i^{in}(w_x, I_2, I_3)$$

and then $0 \leqslant \rho_i(x,t) \leqslant \|f_i^{in}\|_{L^1}$ and thus ρ_i is well-defined and bounded. Now we prove the continuity. Let $(x,t) \in [0,1] \times (-\infty, 0]$ and $(x_n, t_n)_{n\in\mathbb{N}} \subset [0,1] \times (-\infty, 0]$ a sequence such that $\lim\limits_{n\to+\infty} (x_n, t_n) = (x,t)$. Since p_i^0 is continuous one has that for almost every $(w_x, I_2, I_3) \in (0, +\infty) \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2}]$

$$\lim_{n\to+\infty} \frac{f_i^{in}(w_x, I_2, I_3) w_x}{\sqrt{w_x^2 + 2(p_i^0(x_n, I_3) - t_n)}} = \frac{f_i^{in}(w_x, I_2, I_3) w_x}{\sqrt{w_x^2 + 2(p_i^0(x, I_3) - t)}}.$$

Moreover, one has for all $n \in \mathbb{N}$

$$\frac{f_i^{in}(w_x, I_2, I_3) w_x}{\sqrt{w_x^2 + 2(p_i^0(x_n, I_3) - t_n)}} \leqslant f_i^{in}(w_x, I_2, I_3) \text{ for a.e } (w_x, I_2, I_3) \in (0, +\infty) \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2}).$$

Since $f_i^{in} \in L^1$, from the Lebesgue dominated convergence theorem we deduce that $\lim\limits_{n\to+\infty} \rho_i(x_n, t_n) = \rho_i(x,t)$. □

Proposition 4.5.2. *Let $x \in (0,1]$ and $t^*(x) := \frac{-\omega_i^2 x(1-x)}{2\mu} + x\phi_w$. Then the function $\rho_e(x,.)$ is well-defined on $[t^*(x), +\infty)$ and continuous. One has the bound*

$$|\rho_e(x,t)| \leqslant 2\sqrt{2} n_0 (2\alpha + 8).$$

Moreover, for all $t \geqslant t^*(x)$

$$\begin{aligned}
\rho_e(x,t) &= (1+\alpha) n_0 \frac{e^t}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erfc}\left(\sqrt{p_e^1(x, E_3) + t}\right) d\tilde{E}_3 \\
&+ (1+\alpha) n_0 \frac{e^t}{\sqrt{\pi}} \int_{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2} \operatorname{erfc}\left(\sqrt{p_e^0(x, E_3) + t}\right) d\tilde{E}_3 \\
&+ \frac{2n_0 e^t}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erf}\left(\sqrt{p_e^1(x, E_3) + t}\right) d\tilde{E}_3 \\
&- \frac{2n_0 e^t}{\sqrt{\pi}} \int_{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t) - \frac{\omega_i x}{\mu})}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} \operatorname{erf}\left(\sqrt{p_e^0(x, E_3) + t}\right) d\tilde{E}_3.
\end{aligned} \tag{4.58}$$

Proof. We begin with showing that ρ_e is well-defined and bounded. Notice that for all $(x,t) \in (0,1] \times [t^*(x), +\infty)$ one has

$$\begin{aligned}
&\forall E_3 \geqslant E(\phi_w) \quad p_e^0(x, E_3) + t = -\frac{\omega_i^2 x^2}{2\mu} + E_3 \omega_i x + t \geqslant -\frac{\omega_i^2 x^2}{2\mu} + E(\phi_w)\omega_i x + t \\
&\geqslant -\frac{\omega_i^2 x^2}{2\mu} + (\frac{\omega_i^2}{2\mu} - \phi_w) x + t \geqslant -t^*(x) + t \geqslant 0.
\end{aligned}$$

$$\begin{aligned}
&\forall E_3 \leqslant E(\phi_w) \quad p_e^1(x,E_3)+t = \frac{\omega_i^2}{2\mu}(1-x^2)+E_3\omega_i(x-1)-\phi_w+t\\
&\geqslant \frac{\omega_i^2}{2\mu}(1-x^2)+E(\phi_w)\omega_i(x-1)-\phi_w+t\\
&\geqslant \frac{\omega_i^2}{2\mu}(1-x^2)+(\frac{\omega_i^2}{2\mu}-\phi_w)(x-1)-\phi_w+t\\
&\geqslant -t^*(x)+t \geqslant 0.
\end{aligned}$$

The square roots are then well-defined and the bound easily follows from the fact that $|\operatorname{erf}(x)| \leqslant 1$, $|\operatorname{erfc}(x)| \leqslant 2$ for all $x \in \mathbb{R}$ and the fact that $\int_I e^{-\tilde{E_3}^2} d\tilde{E_3} \leqslant \sqrt{\pi}$ for any interval $I \subset \mathbb{R}$. Now we prove that it can be expressed in another way and that it it is continuous. One can decompose the function $\rho_e(x,.)$ as

$$\rho_e(x,t) = \rho_e^1(x,t)+\rho_e^2(x,t)+\rho_e^3(x,t)+\rho_e^4(x,t)+\rho_e^5(x,t)$$

with

$$\begin{aligned}
\rho_e^1(x,t) &:= (1+\alpha)n_0\frac{e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\operatorname{erfc}\left(\sqrt{p_e^1(x,E_3)+t}\right)d\tilde{E_3}\\
\rho_e^2(x,t) &:= (1+\alpha)n_0\frac{e^t}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2}\operatorname{erfc}\left(\sqrt{p_e^0(x,E_3)+t}\right)d\tilde{E_3}\\
\rho_e^3(x,t) &:= 2n_0\mathbf{1}_{\mathcal{E}(x,t)\geqslant E(\phi_w)}(x)\frac{e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\operatorname{erf}\left(\sqrt{p_e^1(x,E_3)+t}\right)d\tilde{E_3}\\
\rho_e^4(x,t) &:= 2n_0\mathbf{1}_{\mathcal{E}(x,t)< E(\phi_w)}(x)\frac{e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\operatorname{erf}\left(\sqrt{p_e^1(x,E_3)+t}\right)d\tilde{E_3}\\
\rho_e^5(x,t) &:= 2n_0\mathbf{1}_{\mathcal{E}(x,t)< E(\phi_w)}(x)\frac{e^t}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t)-\frac{\omega_i x}{\mu})}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\left(\operatorname{erf}\left(\sqrt{p_e^1(x,E_3)+t}\right)-\operatorname{erf}\left(\sqrt{p_e^0(x,E_3)+t}\right)\right)d\tilde{E_3}.
\end{aligned}$$

Remark that $t^*(x)$ is the unique point such that $\mathcal{E}(x,t) = E(\phi_w)$ and $\mathcal{E}(x,t) \geqslant E(\phi_w)$ if and only if $t \leqslant t^*(x)$. Therefore for all $t > t^*(x)$,

$$\begin{aligned}
&\rho_e(x,t) = (1+\alpha)n_0\frac{e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\operatorname{erfc}\left(\sqrt{p_e^1(x,E_3)+t}\right)d\tilde{E_3}\\
&+(1+\alpha)n_0\frac{e^t}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2}\operatorname{erfc}\left(\sqrt{p_e^0(x,E_3)+t}\right)d\tilde{E_3}\\
&+\frac{2n_0e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\operatorname{erf}\left(\sqrt{p_e^1(x,E_3)+t}\right)d\tilde{E_3}\\
&+\frac{2n_0e^t}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t)-\frac{\omega_i x}{\mu})}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\left(\operatorname{erf}\left(\sqrt{p_e^1(x,E_3)+t}\right)-\operatorname{erf}\left(\sqrt{p_e^0(x,E_3)+t}\right)\right)d\tilde{E_3}.
\end{aligned}$$

Then using Chasles relation we obtain

$$\rho_e(x,t) = (1+\alpha)n_0\frac{e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\,\mathrm{erfc}\left(\sqrt{p_e^1(x,E_3)+t}\right)d\tilde{E_3}$$
$$+ (1+\alpha)n_0\frac{e^t}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2}\,\mathrm{erfc}\left(\sqrt{p_e^0(x,E_3)+t}\right)d\tilde{E_3}$$
$$+ \frac{2n_0e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\,\mathrm{erf}\left(\sqrt{p_e^1(x,E_3)+t}\right)d\tilde{E_3}$$
$$- \frac{2n_0e^t}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t)-\frac{\omega_i x}{\mu})}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\,\mathrm{erf}\left(\sqrt{p_e^0(x,E_3)+t}\right)d\tilde{E_3}.$$

and

$$\rho_e(x,t^*(x)) = (1+\alpha)n_0\frac{e^{t^*(x)}}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\,\mathrm{erfc}\left(\sqrt{p_e^1(x,E_3)+t^*(x)}\right)d\tilde{E_3}$$
$$+ (1+\alpha)n_0\frac{e^{t^*(x)}}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2}\,\mathrm{erfc}\left(\sqrt{p_e^0(x,E_3)+t^*(x)}\right)d\tilde{E_3}$$
$$+ \frac{2n_0e^{t^*(x)}}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\,\mathrm{erf}\left(\sqrt{p_e^1(x,E_3)+t^*(x)}\right)d\tilde{E_3}.$$

By standard arguments, one can show that $\rho_e(x,.)$ is continuous on $(t^*(x),+\infty)$. The continuity at $t^*(x)$ also follows from standard arguments and the fact that $\lim\limits_{t\to t^*(x)^+}\mathcal{E}(x,t) = E(\phi_w)$. □

Remark 34. *Note that ρ_e is non negative*

We also need to know whether ρ_i and ρ_e are of class C^1.

Proposition 4.5.3. *Let $f_i^{in}\in L^1\cap L^\infty((0,+\infty)\times\mathbb{R}^2;\mathbb{R}^+)$ such that $\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty}\frac{f_i^{in}(w_x,I_2,I_3)}{w_x^2}dw_xdI_3dI_2 < +\infty$. Then for all $x\in[0,1]$, the function $\rho_i(x,.)$ is of class C^1 on $(-\infty,0]$ with*

$$\frac{\partial\rho_i}{\partial t}(x,t) = \int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty}\frac{f_i^{in}(w_x,I_2,I_3)w_x}{(w_x^2+2(p_i^0(x,I_3)-t))^{\frac{3}{2}}}dw_xdI_3dI_2.$$

Proof. It suffices to apply the theorem of differentiation under the integral sign. The key point is the bound

$$0\leqslant\frac{f_i^{in}(w_x,I_2,I_3)w_x}{(w_x^2+2(p_i^0(x,I_3)-t))^{\frac{3}{2}}}\leqslant\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty}\frac{f_i^{in}(w_x,I_2,I_3)}{w_x^2}dw_xdI_3dI_2$$

for $t\leqslant 0$ and for almost every $(w_x,I_2,I_3)\in(0+\infty)\times\mathbb{R}\times(-\infty,-\frac{\omega_i}{2})$. □

Proposition 4.5.4. *Let $x\in[0,1]$ then $\rho_e(x,.)$ is of class C^1 on $(t^*(x),+\infty)$ with*

$$\frac{\partial\rho_e}{\partial t}(x,t) = \rho_e(x,t)+\kappa_e(x,t)$$

where

$$\kappa_e(x,t) := (1+\alpha)n_0\frac{e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\frac{\mathrm{erfc}'\left(\sqrt{p_e^1(x,E_3)+t}\right)}{2\sqrt{p_e^1(x,E_3)+t}}d\tilde{E}_3$$
$$+ (1+\alpha)n_0\frac{e^t}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})}^{+\infty} e^{-\tilde{E_3}^2}\frac{\mathrm{erfc}'\left(\sqrt{p_e^0(x,E_3)+t}\right)}{2\sqrt{p_e^0(x,E_3)+t}}d\tilde{E}_3$$
$$+ \frac{2n_0e^t}{\sqrt{\pi}}\int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\frac{\mathrm{erf}'\left(\sqrt{p_e^1(x,E_3)+t}\right)}{2\sqrt{p_e^1(x,E_3)+t}}d\tilde{E}_3$$
$$- \frac{2n_0e^t}{\sqrt{\pi}}\int_{\sqrt{\frac{\mu}{2}}(\mathcal{E}(x,t)-\frac{\omega_i x}{\mu})}^{\sqrt{\frac{\mu}{2}}(E(\phi_w)-\frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2}\frac{\mathrm{erf}'\left(\sqrt{p_e^0(x,E_3)+t}\right)}{2\sqrt{p_e^0(x,E_3)+t}}d\tilde{E}_3$$

4.5.2 Functional setting for the (NLP-MMAG) problem

We now set a functional framework in which it is possible to study the (NLP-MMAG) problem. We consider ion incoming boundary conditions that are integrable, essentially bounded and of finite kinetic energy. Because n_0 and ϕ_w are well-determined with the assumption (4.24), we define for any $\omega_i > 0$ the set

$$\mathcal{I}^{in}(\omega_i) := \left\{ f_i^{in} \in (L^1 \cap L^\infty)\left(\mathbb{R}^+ \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2}); \mathbb{R}^+\right) \text{ such that } v_x^2 f_i^{in} \in L^1(\mathbb{R}^+ \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2}) \right\}.$$

The set of admissible ion incoming boundary condition is then defined for all $\alpha \in [0,1)$, $\omega_i > 0$ and $\mu > 0$ by

$$\mathcal{I}_{ad}^{in}(\alpha,\omega_i,\mu) := \left\{ f_i^{in} \in \mathcal{I}^{in}(\omega_i) \text{ such that } 0 < \frac{\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x,I_2,I_3)w_x dw_x dI_3 dI_2}{\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x,I_2,I_3) dw_x dI_3 dI_2} \leqslant s_1(\alpha,\omega_i,\mu) \right\}$$

where $s_1(\alpha,\omega_i,\mu)$ is defined by (4.50). Then for all $f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha,\omega_i,\mu)$ there exists a unique $\phi_w < 0$ solution to (4.47). Let us define the Hilbert space

$$V_0 := \{\phi \in H^1(0,1) \text{ such that } \phi(0) = 0\}$$

equipped with the inner product defined for all $(\varphi,\psi) \in V_0^2$ by $(\varphi,\psi)_{V_0} := \int_0^1 \varphi'(x)\psi'(x)dx$. Let us also define the function

$$\underline{q}(x) := \frac{-\omega_i^2 x(1-x)}{2\mu} + x\phi_w \quad \forall x \in [0,1].$$

In view of Section 4.5.1, we define the set of admissible potentials as

$$V_{ad}(\alpha,\omega_i,\mu) := \left\{\phi \in V_0 \text{ such that } \phi(1) = \phi_w \text{ and } \underline{q}(x) \leqslant \phi(x) \leqslant 0\, \forall x \in [0,1]\right\}.$$

It is a closed and convex subset of V_0. The non linear Poisson problem (NLP-MMAG) reformulates as:

$$\begin{cases} \text{Let } \alpha \in [0,1), \omega_i > 0, \mu > 0, f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha,\omega_i,\mu) \text{ and } \varepsilon > 0, \\ \text{find } \phi \in V_{ad}(\alpha,\omega_i,\mu) \\ -\varepsilon^2\phi''(x) = \rho_i(x,\phi(x)) - \rho_e(x,\phi(x)) \text{ for all } x \in (0,1). \end{cases}$$

where ρ_i and ρ_e are defined by (4.56) and (4.58).

Remark 35. *Note that if there is a solution $\phi \in V_{ad}(\alpha, \omega_i, \mu)$ to (NLP-M), since $H^1(0,1) \hookrightarrow C^0[0,1]$, then the function $x \in (0,1) \mapsto \rho_i(x, \phi(x)) - \rho_e(x, \phi(x))$ is bounded:*

$$|\rho_i(x, \phi(x)) - \rho_e(x, \phi(x))| \leqslant 2\sqrt{2} n_0 (2\alpha + 8) + \|f_i^{in}\|_{L^1}.$$

Therefore $\phi \in W^{2,\infty}(0,1)$ and from the continuous embedding $W^{2,\infty}(0,1) \hookrightarrow C^1[0,1]$ we deduce that $\phi \in C^1[0,1]$ and moreover the function $x \in (0,1] \mapsto \rho_i(x, \phi(x)) - \rho_e(x, \phi(x))$ can be extended by continuity at $x = 0$ with $\rho_i(0, \phi(0)) - \rho_e(0, \phi(0)) = 0$.

4.5.3 The minimization formulation of (NLP-MMAG)

We present a minimization formulation of the problem (NLP-MMAG). Let consider $x \in (0,1]$, since the function $\rho_e(x, .)$ is continuous on $[\underline{q}(x), +\infty)$ we can define for all $t \geqslant \underline{q}(x)$ the function

$$\mathcal{U}_e(x,t) := \int_{\underline{q}(x)}^{t} \rho_e(x,u) du \tag{4.59}$$

where ρ_e is defined by (4.58). It is the unique function such that $\frac{\partial}{\partial t}\mathcal{U}_e(x,t) = \rho_e(x,t)$ and $\mathcal{U}_e(x, \underline{q}(x)) = 0$. Let us also define for all $x \in [0,1]$ and $t \leqslant 0$ the function

$$\mathcal{U}_i(x,t) := -\int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) w_x \sqrt{w_x^2 + 2(p_i^0(x, I_3) - t)} dw_x dI_3 dI_2 \tag{4.60}$$

which is such that $\frac{\partial}{\partial t}\mathcal{U}_i(x,t) = \rho_i(x,t)$. One should remark that solutions to (NLP-MMAG) are critical points of the energy functional defined by

$$J_\varepsilon(\phi) := \int_0^1 \frac{\varepsilon^2}{2} |\phi'(x)|^2 - \mathcal{U}(x, \phi(x)) dx \quad \forall \phi \in V_{ad}(\alpha, \omega_i, \mu). \tag{4.61}$$

where for all $x \in (0,1]$

$$\mathcal{U}(x,t) = \mathcal{U}_i(x,t) - \mathcal{U}_e(x,t) \quad t \in [\underline{q}(x), 0]. \tag{4.62}$$

Also remark that thanks to proposition 4.5.1 and 4.5.2, for all $x \in (0,1]$ the function $\mathcal{U}(x, .)$ is of class C^1 on $[\underline{q}(x), 0]$. We are then going to study the minimization problem:

$$\begin{cases} \text{Let } \alpha \in [0,1), \omega_i > 0, \mu > 0, f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha, \omega_i, \mu) \text{ and } \varepsilon > 0, \\ \text{find } \phi \in V_{ad}(\alpha, \omega_i, \mu) \text{ such that} \\ \phi = \arg \min\limits_{\psi \in V_{ad}(\alpha, \omega_i, \mu)} J_\varepsilon(\psi). \end{cases} \tag{4.63}$$

Consider the Nemytskii operator associated with $\mathcal{U}$ which is denoted $T_{\mathcal{U}}$ and defined for all $\phi \in V_{ad}(\alpha, \omega_i, \mu)$ by $T_{\mathcal{U}}(\phi)(x) := \mathcal{U}(x, \phi(x))$ for all $x \in (0,1]$. Notice that J_ε is made of strictly convex part

$$\phi \in V_{ad}(\alpha, \omega_i, \mu) \mapsto E_\varepsilon(\phi) = \frac{\varepsilon^2}{2} \|\phi\|_{V_0}^2 \tag{4.64}$$

minus a perturbation that is not necessarily convex

$$\phi \in V_{ad}(\alpha, \omega_i, \mu) \mapsto F(\phi) = \int_0^1 T_{\mathcal{U}}(\phi)(x) dx. \tag{4.65}$$

To show that the minimization problem has a solution, we need to study the perturbation $T_{\mathcal{U}}$.

Proposition 4.5.5. *Let $\alpha \in [0,1), \omega_i > 0, \mu > 0$ and $f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha, \omega_i, \mu)$. The mapping $T_{\mathcal{U}} : V_{ad}(\alpha, \omega_i, \mu) \to L^\infty(0,1)$ is well-defined and Fréchet differentiable. Moreover, its Fréchet derivative at $\phi \in V_{ad}(\alpha, \omega_i, \mu)$ is the linear mapping from V_0 to $L^\infty(0,1)$ defined by*

$$dT_{\mathcal{U}}(\phi)(h) = \frac{\partial}{\partial t}\mathcal{U}(x,\phi)h = (\rho_i(x,\phi) - \rho_e(x,\phi))h \quad \forall h \in V_0.$$

Proof. Let us show that $T_{\mathcal{U}}$ is well-defined. For all $\phi \in V_{ad}(\alpha, \omega_i, \mu)$ and for all $x \in (0,1]$ one has

$$|T_{\mathcal{U}}(\phi(x))| = |\mathcal{U}(x, \phi(x))| \leqslant |\mathcal{U}_i(x, \phi(x))| + |\mathcal{U}_e(x, \phi(x))|.$$

Then using propositions 4.5.1 and 4.5.2 we get the bound

$$|T_{\mathcal{U}}(\phi(x))| \leqslant \|v_x^2 f_i^{in}\|_{L^1} + 2\sqrt{2}n_0(2\alpha+8)|\phi(x) - \underline{q}(x)|$$
$$|T_{\mathcal{U}}(\phi(x))| \leqslant \|v_x^2 f_i^{in}\|_{L^1} + 2\sqrt{2}n_0(2\alpha+8)|\underline{q}(x)|.$$

and since $\underline{q}$ is bounded, it shows that $T_{\mathcal{U}}(\phi)$ is essentially bounded. Now we prove that $T_{\mathcal{U}}$ is differentiable on $V_{ad}(\alpha, \omega_i, \mu)$. Let $\phi \in V_{ad}(\alpha, \omega_i, \mu)$ and take $h \in V_0$ such that $\phi + h \in V_{ad}(\alpha, \omega_i, \mu)$. Let $x \in (0,1]$, since $\mathcal{U}(x,.)$ is $C^1[\underline{q}(x), 0]$, by a Taylor expansion we get that there exists a function $\epsilon_x : \mathbb{R} \to \mathbb{R}$ such that $\lim\limits_{t\to 0} \epsilon_x(t) = 0$ and

$$\mathcal{U}(x, \phi(x) + h(x)) = \mathcal{U}(x, \phi(x)) + h(x)\frac{\partial}{\partial t}\mathcal{U}(x, \phi(x)) + h(x)\epsilon_x(h(x)).$$

If $\|h\|_{V_0} \to 0$ then from the continuous embedding $V_0 \hookrightarrow C^0[0,1]$ we deduce that the function h tends uniformly to the zero function and in particular for all $x \in [0,1]$, $h(x) \to 0$ and therefore $\|\epsilon_x(h)\|_{L^\infty(0,1)} \to 0$ as $\|h\|_{V_0} \to 0$. The mapping $T_{\mathcal{U}}$ is therefore differentiable and its Fréchet derivative is the linear mapping $dT_{\mathcal{U}}(\phi) : V_0 \to L^\infty(0,1)$ defined for all $h \in V_0$ by

$$dT_{\mathcal{U}}(\phi)(h) = \frac{\partial}{\partial t}\mathcal{U}(x,\phi) = (\rho_i(x,\phi) - \rho_e(x,\phi))h.$$

It is well-defined and continuous thanks to the bounds of propositions 4.5.1 and 4.5.2. ☐

Proposition 4.5.6. *Let $\alpha \in [0,1), \omega_i > 0, \mu > 0$ and $f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha, \omega_i, \mu)$. The mapping $T_{\mathcal{U}} : V_{ad}(\alpha, \omega_i, \mu) \to L^\infty(0,1)$ is compact.*

Proof. One can decompose the mapping $T_{\mathcal{U}} = \tilde{T_{\mathcal{U}}} \circ i$ where $i : V_{ad}(\alpha, \omega_i, \mu) \to V_{ad}(\alpha, \omega_i, \mu) \cap L^\infty(0,1)$ is the Rellich compact embedding and $\tilde{T_{\mathcal{U}}} : \phi \in L^\infty(0,1) \mapsto \mathcal{U}(x,\phi) \in L^\infty(0,1)$ is continuous. Therefore $T_{\mathcal{U}}$ is compact. ☐

Now we can prove the existence of minimizers.

Theorem 4.5.7. *Let $\alpha \in [0,1), \omega_i > 0, \mu > 0$ and $f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha, \omega_i, \mu)$ and $\varepsilon > 0$. There exists $\phi_\varepsilon \in V_{ad}(\alpha, \omega_i, \mu)$ such that*

$$J_\varepsilon(\phi_\varepsilon) \leqslant J_\varepsilon(\psi) \quad \forall \psi \in V_{ad}(\alpha, \omega_i, \mu).$$

Moreover, one has the estimate

$$\|\phi_\varepsilon\|_{V_0} = \mathcal{O}(\frac{1}{\varepsilon}).$$

Proof. It suffices to verify that J_ε is coercive on the closed and convex subset $V_{ad}(\alpha, \omega_i, \mu)$ and weakly-lower-semi-continuous. To do so, first remark that for all $\phi \in V_{ad}(\alpha, \omega_i, \mu)$ one has

$$J_\varepsilon(\phi) = E_\varepsilon(\phi) + F(\phi)$$

and note that one has the bound

$$|F(\phi)| \leqslant \|v_x^2 f_i^{in}\|_{L^1} + 2\sqrt{2} n_0 (2\alpha + 8) \int_0^1 |\underline{q}(x)| dx.$$

Therefore

$$J_\varepsilon(\phi) \geqslant \frac{\varepsilon^2}{2} \|\phi\|_{V_0}^2 - (\|v_x^2 f_i^{in}\|_{L^1} + 2\sqrt{2} n_0 (2\alpha + 8) \int_0^1 |\underline{q}(x)| dx)$$

and by comparison we get that $J_\varepsilon(\phi) \to +\infty$ as $\|\phi\|_{V_0} \to +\infty$, and thus it is coercive. We now prove that J_ε is weakly lower semi-continuous. Note that E_ε is convex and continuous for the strong topology of V_0 and therefore it is weakly lower semi-continuous. The mapping F is compact from $V_{ad}(\alpha, \omega_i, \mu)$ to $\mathbb{R}$ and it is therefore weakly lower semicontinuous. Thus there exits $\phi_\varepsilon \in V_{ad}(\alpha, \omega_i, \mu)$ such that $J_\varepsilon(\phi_\varepsilon) \leqslant J_\varepsilon(\psi)$ for all $\psi \in V_{ad}(\alpha, \omega_i, \mu)$. We prove the estimate. Since $\phi_w < 0$ does not depend on ε we have by taking $x \in [0,1] \mapsto \psi(x) = x\phi_w \in V_{ad}(\alpha, \omega_i, \mu)$ as a test function

$$\frac{\varepsilon^2}{2} \|\phi\|_{V_0}^2 - (\|v_x^2 f_i^{in}\|_{L^1} + 2\sqrt{2} n_0 (2\alpha + 8) \int_0^1 |\underline{q}(x)| dx) \leqslant J_\varepsilon(\phi_\varepsilon) \leqslant J_\varepsilon(x\phi_w),$$

therefore

$$\frac{\varepsilon^2}{2} \|\phi\|_{V_0}^2 \leqslant \frac{\varepsilon^2}{2} \phi_w^2 + \left(\int_0^1 \mathcal{U}(x, x\phi_w) dx + \|v_x^2 f_i^{in}\|_{L^1} + 2\sqrt{2} n_0 (2\alpha + 8) \int_0^1 |\underline{q}(x)| dx \right)$$

and dividing by $\frac{\varepsilon^2}{2}$ each side of the inequality yields the result. $\square$

Minimizers of J_ε are not necessarily critical points since $V_{ad}(\alpha, \omega_i, \mu)$ is a strict and closed convex subset of V_0. In particular a minimizer could belong to the boundary $\partial V_{ad}(\alpha, \omega_i, \mu)$ which is the set of functions belonging to $V_{ad}(\alpha, \omega_i, \mu)$ that either vanish on some non empty interval of $(0,1)$ or coincide with $\underline{q}$ on some nom empty interval of $(0,1)$. We can however characterize the minimizers following the same idea as in [61].

Proposition 4.5.8 (First order condition)**.** *Let $\alpha \in [0,1)$,$\omega_i > 0$, $\mu > 0$ and $f_i^{in} \in \mathcal{I}_{ad}(\alpha, \omega_i, \mu)$ and $\varepsilon > 0$. Let $\phi := \phi_\varepsilon \in V_{ad}(\alpha, \omega_i, \mu)$ be a minimizer of J_ε. Then the following variational inequality holds*

$$dJ_\varepsilon(\phi)(h) \geqslant 0 \quad \text{for all } h \in V_0 \text{ such that } \phi + h \in V_{ad}(\alpha, \omega_i, \mu). \tag{4.66}$$

Moreover, we have $\phi \in W^{2,\infty}(0,1) \cap C^1[0,1]$ and

$$-\varepsilon^2 \phi''(x) = \frac{\partial}{\partial t} \mathcal{U}(x, \phi(x)) \quad \text{a.e in } \mathcal{O} := \{x \in (0,1) \mid \underline{q}(x) < \phi(x) < 0\}, \tag{4.67}$$

$$-\varepsilon^2 \phi''(x) \leqslant \frac{\partial}{\partial t} \mathcal{U}(x, \phi(x)) \quad \text{a.e in } \mathcal{F}_1 := \{x \in (0,1) \mid \phi(x) = 0\}, \tag{4.68}$$

$$-\varepsilon^2 \phi''(x) \geqslant \frac{\partial}{\partial t} \mathcal{U}(x, \phi(x)) \quad \text{a.e in } \mathcal{F}_2 := \{x \in (0,1) \mid \phi(x) = \underline{q}(x)\}. \tag{4.69}$$

Let us finish this section with a list of insightful remarks about the range of applicability of this result and the difficulties that underly the solvability of the (NLP-MMAG) problem.

Remark 36. *It is not guaranteed that our minimization technique produces a solution to (NLP-MMAG). The reason is easily seen from the proposition (4.5.8). Indeed, a minimizer to J_ε that belongs to $\partial V_{ad}(\alpha,\omega_i,\mu)$ does not necessarily cancel the gradient of J_ε. The first order condition is in that case an inequality and traduces the fact that a minimizer is sub and super solutions of (NLP-MMAG) on some non empty intervals of $(0,1)$.*

Remark 37. *To enforce a minimizer to be a solution, one could seek for conditions under which the inequalities in the proposition (4.5.8) are equalities. When ε is small enough, the set $\mathcal{F}_1$ of proposition (4.5.8) is non-empty. It therefore seems to be a key point to be able to make the inequality becomes an equality. On the contrary to the purely electrostatic case ($\omega_i = 0$) for which we were able to so, it seems to be much more intricate in the presence of a magnetic field. The idea is to look for ion boundary condition such that for all $x \in \mathcal{F}_1$, $\frac{\partial}{\partial t}\mathcal{U}(x,0) = 0$ which is equivalent to $\rho_i(x,0) = \rho_e(x,0)$ for all $x \in \mathcal{F}_1$. We have not been able to identify simple conditions on f_i^{in} that enables to do so.*

Remark 38. *One has already $(\rho_i - \rho_e)(0,0) = 0$ and for a minimizer ϕ_ε, one has $\phi_\varepsilon(0) = 0$. A natural idea is therefore to make $\rho_i(x,0) = \rho_e(x,0)$ constant in a vicinity of $x = 0$. One can seek for f_i^{in} being such that $\frac{d^k}{dx^k}(\rho_i - \rho_e)(0,0) = 0$ for a range of $k \geqslant 1$.*

Remark 39. *There is no hope to make $\rho_i(.,0)$ and $\rho_e(.,0)$ constant independently in a vicinity of $x = 0$. Indeed a quick computation shows that for ion boundary conditions that are sufficiently integrable one has*

$$\frac{d}{dx}\rho_i(0,0) = \omega_i \int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} \frac{f_i^{in}(w_x, I_2, I_3) I_3}{w_x^2} dw_x dI_3 dI_2 < 0.$$

Therefore, $\rho_i(.,0)$ cannot be constant in a neighbourhood of $x = 0$.

4.6 Numerical approximation of (NLP-MMAG)

We are now going to describe the numerical methods we employ to approximate solutions of (NLP-MMAG) via our minimization technique. We also explain how we identify in practice a range of parameters that give reasonable solution. We finally give some illustrations of the results we obtain and draw a parallel with the results obtained by Manfredi and Coulette in [43]. All the numerical methods described here can be considered as standard tools. Our goal here is not to justify the approximations but only to give an idea on how the numerical methods have been implemented. Note that to solve (NLP-MMAG), one has first of all to solve the non linear equation (4.47). The numerical approximation of solutions to (NLP-MMAG) is naturally in two independent step. Namely given $\alpha \in [0,1)$, $\omega_i > 0$, $\mu > 0$and $f_i^{in} \in \mathcal{I}_{in}^{ad}(\alpha,\omega_i,\mu)$:

1. we need to approximate the wall potential ϕ_w solution to (4.47) and construct an approximation of n_0.
2. given $\varepsilon > 0$, we approximate ϕ_ε minimizing the functional J_ε on $V_{ad}(\alpha,\omega_i,\mu)$.

4.6.1 Description of the numerical methods

Numerical quadrature for velocity integrals

Both step of the numerical approximation require to compute velocity integrals. We present the way we proceed numerically to compute them. For a target function $g : [0,+\infty) \times (-\infty, -\frac{\omega_i}{2}] \times \mathbb{R} \to \mathbb{R}^+$ that is

L^1 and smooth we define:

$$I(g) := \int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} g(v_x, v_y, v_z) dv_x dv_z dv_y = \int_{\mathbb{R}} \int_{-\infty}^{0} \int_0^{+\infty} g(v_x, v_y, \tilde{v_z} - \frac{\omega_i}{2}) dv_x d\tilde{v_z} dv_y. \tag{4.70}$$

To avoid an unnecessary numerical truncation of the integration domain we decide to use spherical coordinates:

$$\begin{cases} v_x = v_r \cos(u)\cos(\theta) \\ v_y = v_r \cos(u)\sin(\theta) \\ \tilde{v_z} = v_r \sin(u) \\ v_r = \sqrt{v_x^2 + v_y^2 + \tilde{v_z}^2}, \end{cases}$$

where $(u, \theta) \in [-\frac{\pi}{2}, 0] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. One has therefore

$$I(g) = \int_0^{+\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} g(v_r \cos(u)\cos(\theta), v_r \cos(u)\sin(\theta), v_r \sin(u) - \frac{\omega_i}{2}) v_r^2 \cos(u) du d\theta dv_r.$$

We split the integral as

$$I(g) = \int_0^{v_r^{cut}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} g(v_r \cos(u)\cos(\theta), v_r \cos(u)\sin(\theta), v_r \sin(u) - \frac{\omega_i}{2}) v_r^2 \cos(u) du d\theta dv_r$$
$$+ \int_{v_r^{cut}}^{+\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} g(v_r \cos(u)\cos(\theta), v_r \cos(u)\sin(\theta), v_r \sin(u) - \frac{\omega_i}{2}) v_r^2 \cos(u) du d\theta dv_r$$

where $v_r^{cut} > 0$ is chosen such that g is smaller than 10^{-6} on the complementary set of the semi-ball

$$B_{-\frac{\omega_i}{2}}(v_r^{cut}) = \{(v_x, v_y, v_z) \in [0, +\infty) \times (-\infty, -\frac{\omega_i}{2}] \times \mathbb{R} \text{ such that } \sqrt{v_x^2 + v_y^2 + (v_z + \frac{\omega_i}{2})^2} \leqslant v_r^{cut}\}.$$

We then approximate each integrals with a quadrature formula. More precisely, let

$$I_1(g) := \int_0^{v_r^{cut}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} g(v_r \cos(u)\cos(\theta), v_r \cos(u)\sin(\theta), v_r \sin(u) - \frac{\omega_i}{2}) v_r^2 \cos(u) du d\theta dv_r,$$
$$I_2(g) := \int_{v_r^{cut}}^{+\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} g(v_r \cos(u)\cos(\theta), v_r \cos(u)\sin(\theta), v_r \sin(u) - \frac{\omega_i}{2}) v_r^2 \cos(u) du d\theta dv_r.$$

We approximate $I_1(g)$ with a Gauss-Legendre quadrature. Namely given a quadrature $(w_j, \xi_j)_{j=1 \dots N_g} \subset [0, +\infty) \times (-1, 1)$ or order $2N_g - 1$ with $N_g \in \mathbb{N}^*$ we define for all $(j, k, l) \in \{1, ..., N_g\}^3$

$$\begin{cases} v_r^j = \dfrac{v_r^{cut}(1+\xi_j)}{2}, \\ \theta^k = -\dfrac{\pi(1-\xi_k)}{4} + \dfrac{\pi(1+\xi_k)}{4}, \\ u^l = -\dfrac{\pi(1-\xi_l)}{4}, \end{cases}$$

we then use the following approximation

$$I_1(g) \approx \frac{v_r^{cut}\pi^2}{16} \sum_{(j,k,l)\in\{1,\dots,N_g\}^3} w_j w_k w_l g(v_r^j \cos(u^l)\cos(\theta^k), v_r^j \cos(u^l)\sin(\theta^k), v_r^j \sin(u^l) - \frac{\omega_i}{2})(v_r^j)^2 \cos(u^l).$$

One could improve this approximation by subdividing the domain $[0, v_r^{cut}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ into several subdomains and using a quadrature formula on each subdomain. We approximate $I_2(g)$ using a Gauss-Laguerre quadrature. Namely given a quadrature $(w_j, \xi_j)_{j=1\dots N_g} \subset [0, +\infty) \times [0, +\infty)$ or order $2N_g - 1$ with $N_g \in \mathbb{N}^*$. We also define for all $(j, k, l) \in \{1, \dots, N_g\}^3$

$$\begin{cases} v_r^j = \xi_j + v_r^{cut}, \\ \theta^k = -\dfrac{\pi(1-\xi_k)}{4} + \dfrac{\pi(1+\xi_k)}{4}, \\ u^l = -\dfrac{\pi(1-\xi_l)}{4}, \end{cases}$$

we then use the following approximation

$$I_2(g) \approx \frac{\pi^2}{8} \sum_{(j,k,l)\in\{1,\dots,N_g\}^3} w_j w_k w_l g(v_r^j \cos(u^l)\cos(\theta^k), v_r^j \cos(u^l)\sin(\theta^k), v_r^j \sin(u^l) - \frac{\omega_i}{2})(v_r^j)^2 \exp(\xi_j) \cos(u^l).$$

For quantitative error estimates we refer the reader to [47]. We now present the way we proceed to approximate an integral of the form

$$\rho(x,t) := \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} g(x, t, E_3, \omega_i) d\tilde{E}_3 \text{ with } E_3 = \sqrt{\frac{2}{\mu}}\tilde{E}_3 + \frac{\omega_i x}{\mu}$$

and where $g : [0,1] \times \mathbb{R} \times \mathbb{R} \times [0, +\infty)$ is a bounded function such that $g(x, t, E_3, 0) = g(x, t, 0, 0)$ for all $(x, t, E_3) \in [0,1] \times \mathbb{R}^2$. Also remark that since $\phi_w < 0$ is fixed, the upper bound of the integral is such that $\lim_{\omega_i \to 0^+} E(\phi_w) = +\infty$ where $E(\phi_w) = \frac{1}{\omega_i}\left(\frac{\omega_i^2}{2\mu} - \phi_w\right)$. Note that one can prove that for all $(x,t) \in [0,1] \times \mathbb{R}$,

$$\lim_{\omega_i \to 0^+} \rho(x,t) = g(x,t,0,0) \int_{\mathbb{R}} e^{-\tilde{E_3}^2} d\tilde{E}_3 = \sqrt{\pi} g(x,t,0,0).$$

For all $(x,t) \in [0,1] \times \mathbb{R}$, we decompose $\rho(x,t)$ as follows

$$\begin{aligned} \rho(x,t) &= \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} (g(x,t,E_3,\omega_i) - g(x,t,0,0)) d\tilde{E}_3 \\ &\quad + g(x,t,0,0) \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} d\tilde{E}_3 \\ &= \int_{-\infty}^{\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu})} e^{-\tilde{E_3}^2} (g(x,t,E_3,\omega_i) - g(x,t,0,0)) d\tilde{E}_3 \\ &\quad + g(x,t,0,0)\sqrt{\pi}(1 - \frac{1}{2}\operatorname{erfc}(\sqrt{\frac{\mu}{2}}(E(\phi_w) - \frac{\omega_i x}{\mu}))). \end{aligned}$$

We then approximate the first integral with a Gauss-Laguerre quadrature.

Numerical approximation of the wall (floating) potential

The numerical approximation of the wall (floating) potential ϕ_w is done through a standard Newton algorithm. Indeed, given $\alpha \in [0,1)$,$\omega_i > 0$, $\mu > 0$ $f_i^{in} \in \mathcal{I}_{ad}^{in}(\alpha,\omega_i,\mu)$, we consider the problem of finding the unique root of the function defined for all $\psi \leqslant 0$ by

$$\tilde{\mathcal{W}}(\psi) = \mathcal{W}(\psi) - (1+\alpha)\int_{\mathbb{R}}\int_{-\infty}^{-\frac{\omega_i}{2}}\int_0^{+\infty} f_i^{in}(w_x, I_2, I_3) w_x dw_x dI_3 dI_2$$

where $\mathcal{W}$ is the function defined in (4.48). Since $\tilde{\mathcal{W}}' > 0$ on $(-\infty, 0]$ the Newton procedure is well-defined and we can define the sequence $(\phi_w^n)_{n\in\mathbb{N}} \subset (-\infty, 0)$ by induction

$$\begin{cases} \phi_w^0 \leqslant 0, \\ \phi_w^{n+1} = \phi_w^n - \frac{\tilde{\mathcal{W}}(\phi_w^n)}{\tilde{\mathcal{W}}'(\phi_w^n)}, \forall n \in \mathbb{N}. \end{cases}$$

Moreover, since $\mathcal{W}$ is $C^2(-\infty, 0)$, for ϕ_w^0 well chosen this sequence converges to ϕ_w as $n \to +\infty$ [47] (in practice we start with $\phi_w^0 = 0$). The approximation of ϕ_w consists in replacing $\tilde{\mathcal{W}}$ by its numerical approximation with velocity quadrature formulas denoted $I(\tilde{\mathcal{W}})$ and applying the previous Newton algorithm to it. The algorithm is stopped when the last computed term of the sequence is considered as a good approximation of the root of $I(\tilde{\mathcal{W}})$, that is for $\delta > 0$ small enough ($\delta = 10^{-8}$ in the routine) the Newton procedure is stopped when there is $N_\delta \in \mathbb{N}$ such that $|I(\tilde{\mathcal{W}})(\phi_w^{N_\delta})| \leqslant \delta$, the number $\phi_w^{N_\delta}$ is then considered as an acceptable approximation of ϕ_w. The reference density n_0 defined by (4.46) is then approximated by replacing the exact floating potential by $\phi_w^{N_\delta}$, the velocity integrals are approximated by quadrature formulas as described above. The approximation of n_0 is denoted $\tilde{n}_0$.

The projected gradient method for (NLP-MMAG)

Let $\alpha \in [0,1)$, $\omega_i > 0$, $\mu > 0$ and $f_i^{in} \in \mathcal{I}_{ad}(\alpha,\omega_i,\mu)$ and $\varepsilon > 0$. We assume an approximation of the floating potential solution of (4.47) still denoted ϕ_w for the sake on conciseness. To compute numerically the solution to (NLP-MMAG), we lift the boundary conditions and define $\bar{\phi} = \phi - x\phi_w$ where ϕ is solution to (NLP-MMAG) where the floating potential is replaced by its approximation. We therefore consider the equivalent to (NLP-MMAG) Poisson problem

$$\begin{cases} -\frac{\varepsilon^2}{2}\frac{d}{dx^2}\bar{\phi}(x) = \frac{\partial}{\partial t}\bar{\mathcal{U}}(x,\bar{\phi}(x)) \quad \forall x \in (0,1) \\ \bar{\phi}(0) = 0 \text{ and } \bar{\phi}(1) = 0 \\ \text{ with } \bar{\mathcal{U}}(x,t) := \mathcal{U}(x, t + x\phi_w) \text{ for all } x \in [0,1] \text{ and } t \in [\underline{q}(x) - x\phi_w, -x\phi_w]. \end{cases}$$

Then we look for an approximation of minimizers of the functional defined for all $\bar{\phi} \in W = W(\alpha,\omega_i,\mu) := \left\{\bar{\phi} \in H_0^1(0,1) \mid \underline{q}(x) - x\phi_w \leqslant \bar{\phi} \leqslant -x\phi_w \text{ in } [0,1]\right\}$ by

$$\bar{J}_\varepsilon(\bar{\phi}) = \int_0^1 \left(\frac{\varepsilon^2}{2}|\bar{\phi}'(x)|^2 - \bar{\mathcal{U}}(x,\bar{\phi}(x))\right) dx.$$

Let $N \in \mathbb{N}^*$, the discretization consists of a mesh $(x_i := \frac{i}{(N+1)})_{i=0,\ldots,N+1}$ with $h = \frac{1}{N+1}$ and the approximation of the Hilbert space $H_0^1(0,1)$ by a standard and conformous $\mathbb{P}_1$ finite element space V_0^h.

More precisely

$$V_0^h := \left\{ \bar{\phi}_h \in C^0[0,1], \quad \bar{\phi}_h(0) = \bar{\phi}_h(1) = 0 \,|\, \forall i = 0, ..., N \quad \bar{\phi}_h \underset{|[x_i, x_{i+1}]}{\in} \mathbb{P}_1, \right\} \tag{4.71}$$

and the admissible potential set is approximated by

$$W^h := \left\{ \bar{\phi}_h \in V_0^h \,|\, \underline{q}(x) - x\phi_w \leqslant \bar{\phi}_h \leqslant -x\phi_w \text{ in } [0,1] \right\}. \tag{4.72}$$

The discrete minimization problem then writes

$$\begin{cases} \text{Find } \bar{\phi}_h \in W^h \text{ such that} \\ \bar{\phi}_h = \arg \inf\limits_{\psi_h \in W^h} \bar{J}_\varepsilon(\psi_h). \end{cases}$$

This discrete minimization problem has a solution. To approximate its minimizers, we use the projected gradient method. Namely, given $\eta > 0$ and $\delta > 0$, we compute iteratively

$$\begin{cases} \bar{\phi}_h^0 \in W^h \\ \bar{\phi}_h^{n+1} = \Pi\left(\bar{\phi}_h^n - \eta \nabla \bar{J}_\varepsilon(\bar{\phi}_h^n)\right) \end{cases}$$

where $\Pi : V_0^h \to W^h$ is the projection on W^h for the $H_0^1(0,1)$ norm and $\nabla \bar{J}(\bar{\phi}_h^n) \in V_0^h$ is the gradient of $\bar{J}$ at $\bar{\phi}_h^n$, that is the unique solution of the variational problem

$$\begin{cases} \text{Find } u_h \in V_0^h \text{ such that for all } \psi_h \in V_0^h \\ (u_h, \psi_h)_{V_0} = d\bar{J}_\varepsilon(\bar{\phi}_h^n)(\psi_h) \\ d\bar{J}_\varepsilon(\bar{\phi}_h^n)(\psi_h) = \displaystyle\int_0^1 \left(\varepsilon^2 \frac{d}{dx}\bar{\phi}_h^n(x) \frac{d}{dx}\psi_h(x) + \frac{d}{d\psi}\bar{\mathcal{U}}(x, \bar{\phi}_h^n(x))\psi_h(x) \right) dx. \end{cases}$$

The algorithm is known to converge under C^1 regularity and convexity assumption on $\bar{J}$. At this stage, it is not clear whether these conditions are fulfilled. The reader eager to know more about the quantitative error analysis and the proof of convergence of the gradient projected method is referred to [27].

4.6.2 Bounds on the parameters ω_i, ε and applicability domain

However we cannot prove that (NLP-MMAG) is well-posed, it is instructive to illustrate some numerical results for a range of parameters that seems to give consistent solutions of (NLP-MMAG). We will relate these results with those obtained in [43, 48]. Here, we explain how we identify some bounds on the parameters ω_i, ε. These bounds correspond to a domain of parameters for which our algorithms give reasonable solutions, in the sense that the discrete potential ϕ_ε^h solves numerically (NLP-MMAG). The way to check if it does, is to compute the norm of the gradient of $\bar{J}_\varepsilon$ and to consider that it is a reasonable solution when, its norm is lower than 10^{-6}. Let us now describe the steps we follow to identify the bounds:

Bounds on ω_i: We start from the case $\omega_i = 0$ (that we already studied in chapter 2) and fix an incoming distribution function $f_i^{in} \in \mathcal{I}_{ad}(\alpha, 0, \mu)$. Then we find numerically that there is some $\omega_i^c(f_i^{in}) > 0$ such that for all $\omega_i \in [0, \omega_i^c]$, $f_i^{in} \in \mathcal{I}_{ad}(\alpha, \omega_i, \mu)$ and for $\omega_i > \omega_i^c$, $f_i^{in} \notin \mathcal{I}_{ad}(\alpha, \omega_i, \mu)$. The wall potential is defined as long as f_i^{in} satisfies the inequality (4.49). The fact that for $\omega_i > \omega_i^c$, $f_i^{in} \notin \mathcal{I}_{ad}(\alpha, \omega_i, \mu)$ means it is no longer possible to determine a non positive wall potential. The case $\omega_i = \omega_i^c$ corresponds to the equality in the inequality (4.49) and yields the wall potential $\phi_w = 0$.

Bounds on ε: For each $\omega_i \in [0, \omega_i^c(f_i^{in})]$ that determines the associated potential ϕ_w and the reference density n_0

independently of ε, we compute, via the discrete projected gradient algorithm (for a fixed mesh size $h = 2^{-9}$), the discrete electrostatic potential ϕ_ε^h for various $\varepsilon > 0$. We found numerically that there is $\varepsilon^c(\omega_i) > 0$ such that for $\varepsilon \geqslant \varepsilon^c(\omega_i)$, the gradient projected method stopped with a discrete potential that gives $\|\nabla \bar{J}_\varepsilon(\phi_\varepsilon^h)\|_{H_0^1(0,1)} < 10^{-6}$ while for $\varepsilon < \varepsilon^c(\omega_i)$ it stopped with $\|\nabla \bar{J}_\varepsilon(\phi_\varepsilon^h)\|_{H_0^1(0,1)} > 10^{-2}$. It seems that $\varepsilon^c(\omega_i)$ is an increasing function of ω_i and thus $\varepsilon^c(0) \leqslant \varepsilon^c(\omega_i) \leqslant \varepsilon^c(\omega_i^c)$.

Applicability domain: The domain of applicability is then defined for all $f_i^{in} \in \mathcal{I}_{ad}(\alpha, 0, \mu)$ as

$$\mathcal{A}(f_i^{in}) := \bigcup_{0 \leqslant \omega_i \leqslant \omega_i^c(f_i^{in})} \{\omega_i\} \times [\varepsilon^c(\omega_i), +\infty). \tag{4.73}$$

where the parameters $\varepsilon^c(\omega_i)$ is determined as described in the previous step. The applicability domain can be sketched as in figure 4.3.

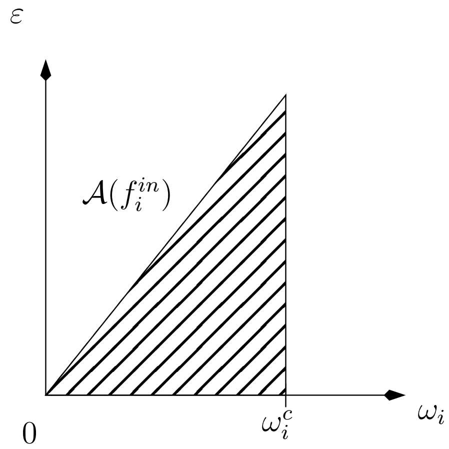


Figure 4.3 – Sketch of the admissibility domain for a given incoming ions boundary condition f_i^{in}. It corresponds to the white area.

Remark 40. *Note that the applicability domain depends on the ion boundary condition f_i^{in}. We think it is possible to enlarge the interval $[0, \omega_i^c(f_i^{in})]$ by considering boundary condition f_i^{in} such that the inequality* (4.49) *holds even with relatively large ω_i.*

4.6.3 Numerical results

In the sequel, we consider a Deuterium plasma that is $\mu = \frac{1}{3672}$ and fix the re-emission parameter $\alpha = 0$. We shall consider two ions boundary conditions, these boundary conditions are inspired from the purely electrostatic case (see chapter 2) and from [43]. For each boundary condition $f_i^{in} \in \mathcal{I}_{ad}(\alpha, 0, \mu)$ corresponds an applicability domain $\mathcal{A}(f_i^{in})$ defined as above (4.73). We define for all $(\omega_i, \varepsilon) \in \mathcal{A}(f_i^{in})$ the ion mean Larmor radius of the incoming ions by

$$r_i^{in} := \frac{\int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} f_i^{in}(v_x, v_y, v_z) \sqrt{v_x^2 + v_z^2}\, dv_x dv_y dv_z}{\omega_i \int_{\mathbb{R}} \int_{-\infty}^{-\frac{\omega_i}{2}} \int_0^{+\infty} f_i^{in}(v_x, v_y, v_z)\, dv_x dv_y dv_z}, \tag{4.74}$$

as well as the ratio between the plasma ion gyrofrequency denoted ω_{ci} and the plasma ion frequency denoted ω_{pi} by

$$\frac{\omega_{ci}}{\omega_{pi}} = \frac{\omega_i \varepsilon}{\sqrt{n_0(\omega_i)}} \text{ where } n_0(\omega_i) > 0 \text{ is given by (4.46).} \tag{4.75}$$

We shall represent the dependency with respect to $\omega_i \in [0, \omega_i^c]$ of the wall potential ϕ_w and the charge at the wall $\rho_i(1, \phi_w) - \rho_e(1, \phi_w)$. These quantities are independent of the normalized Debye length ε. Then for three different values of the magnetic field, $0 \leqslant \omega_i^1 < \omega_i^2 < \omega_i^3 \leqslant \omega_i^c$, we shall represent $x \in [0,1] \mapsto \phi_\varepsilon^h$ a minimizer of J_ε, the charge density $x \in [0,1] \mapsto \rho_i(x, \phi_\varepsilon^h(x)) - \rho_e(x, \phi_\varepsilon^h(x))$ for $\varepsilon = \varepsilon^c(\omega_i^3)$. These values of the parameters ω_i and ε are in the applicability domain. Therefore, the numerical solutions can be considered as consistent with respect to the (NLP-MMAG) problem.

We shall finish with an illustration of some trajectories of the particles in the physical 3d space. In particular, we plot the trajectories for second boundary condition and the couple of parameters $(\omega_i^3, \varepsilon^c(\omega_i^3))$. The results for smaller value of the magnetic field ω_i essentially results in larger ions and electrons Lamor radii and are thus not represented. Our interest here, is above all, to observe the different time scales between the ions and the electrons, and to illustrate the different scenarios predicted in the linear Vlasov study. Notably, an electron that enters at $x = 0$ with $v_z \geqslant E(\phi_w)$ will reach the wall, while an electron with $v_z < E(\phi_w)$ can or cannot reach the wall depending on its v_x velocity. As for the ions, since in our model they are all assumed to enter at $x = 0$ with $v_z < \frac{\omega_i}{2}$ they are all expected to reach $x = 1$. To compute the trajectories, we solve numerically (with a standard explicit Euler-Scheme) the two systems:

$$\text{(Ions):} \begin{cases} \dot{\mathcal{X}}(t) = \mathcal{V}_x(t), \\ \dot{\mathcal{Y}}(t) = \mathcal{V}_y(t), \\ \dot{\mathcal{Z}}(t) = \mathcal{V}_z(t), \\ \dot{\mathcal{V}}_x(t) = -\frac{d}{dx}\phi_\varepsilon^h(\mathcal{X}(t)) - \omega_i \mathcal{V}_z(t), \\ \dot{\mathcal{V}}_y(t) = 0, \\ \dot{\mathcal{V}}_z(t) = \omega_i \mathcal{V}_x(t). \end{cases} \quad \text{(Electrons):} \begin{cases} \dot{\mathcal{X}}(t) = \mathcal{V}_x(t), \\ \dot{\mathcal{Y}}(t) = \mathcal{V}_y(t), \\ \dot{\mathcal{Z}}(t) = \mathcal{V}_z(t), \\ \dot{\mathcal{V}}_x(t) = -\frac{1}{\mu}\left(-\frac{d}{dx}\phi_\varepsilon^h(\mathcal{X}(t)) - \omega_i \mathcal{V}_z(t)\right), \\ \dot{\mathcal{V}}_y(t) = 0, \\ \dot{\mathcal{V}}_z(t) = -\frac{\omega_i}{\mu}\mathcal{V}_x(t), \end{cases}$$

for an initial data at $t = 0$ starting from a point $(x, y, z, v_x, v_y, v_z) \in [0,1] \times \mathbb{R}^5$.

First test case: perturbation of the electrostatic sheath

This test case is inspired from the purely electrostatic case. We consider an ion incoming boundary condition f_i^{in} of the form

$$f_i^{in}(v_x, v_y, v_z) = \min(1, \frac{v_x^2}{\eta}) \frac{e^{-\frac{(v_x - u_x)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} g_i^{in}(v_y) h_i^{in}(v_z), \tag{4.76}$$

for $(v_x, v_y, v_z) \in (0 + \infty) \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2})$, with

$$\eta = 10^{-1}, \quad u_x = 1.5, \quad \sigma = \frac{1}{2}$$
$$g_i^{in}(v_y) = \frac{e^{-\frac{(v_y)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}, \quad h_i^{in}(v_z) = 2\frac{e^{-\frac{(v_z)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}.$$

This boundary condition is not exactly the same as in the purely electrostatic case since it encodes the presence of particles with v_y and v_z velocities. The bounds on the value of the magnetic field is approximately $\omega_i^c(f_i^{in}) \approx 0.06$. In table 4.1, we summarize the different parameters considered for the numerical simulations.

ω_i	ε	$\frac{\omega_{ci}}{\omega_{pi}}$	r_i^{in}
$\omega_i^1 = 0$	0.08	0	$+\infty$
$\omega_i^2 = 0.025$	0.08	$1.7E^{-2}$	77.1
$\omega_i^3 = 0.05$	0.08	$3.5E^{-2}$	38.6

Table 4.1 – Values of the different parameters for the first boundary condition f_i^{in} given by (4.76).

In figure 4.4, we represent the value of the wall potential ϕ_w solution to (4.47). In figure 4.5, we represent the charge at the wall $n_i(1) - n_e(1) = \rho_i(1, \phi_w) - \rho_e(1, \phi_w)$ as functions of ω_i. The definition domain of these functions is the interval $[0, \omega_i^c(f_{in})]$ where $\omega_i^c(f_i^{in}) > 0$ is such that the inequality (4.49) is an equality. Its value is approximately 0.06. We observe that the wall potential seems to be a non decreasing function. The charge at the wall seems positive and non decreasing. As the intensity of the magnetic field increases the wall potential in absolute value decreases and the charge at the wall increases.
In figure 4.6 and 4.7, we represent the electrostatic potential and the charge density for the three different

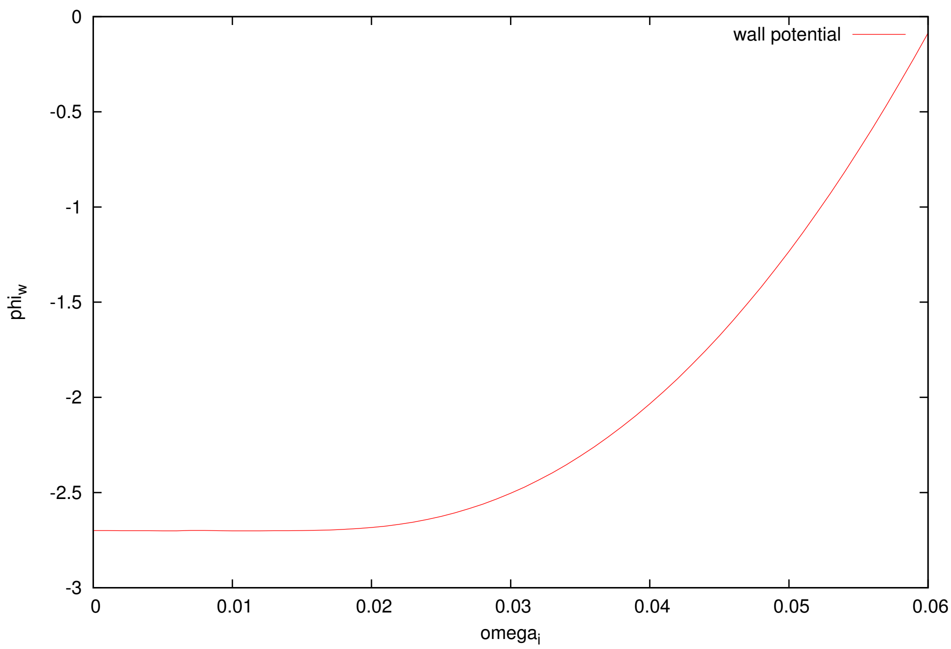


Figure 4.4 – Graph of the function $\omega_i \mapsto \phi_w$ where ϕ_w is the solution to (4.47) for the first test case. Here the critical value ω_i^c is slightly larger than 0.06 for which $\phi_w = 0$.

values of the magnetic field : $w_i^1 = 0$, $\omega_i^2 = 0.025$ and $\omega_i^3 = 0.05$. The corresponding critical value of the normalized Debye length for ω_i^3 is approximately $\varepsilon^c(\omega_i^3) \approx 0.8$. We see on figure 4.7 that increasing the magnetic field results in the charge density being everywhere increased except at $x = 0$ where the neutrality holds. The neutrality breaks when $x > 0$ and the charge is non negative everywhere.

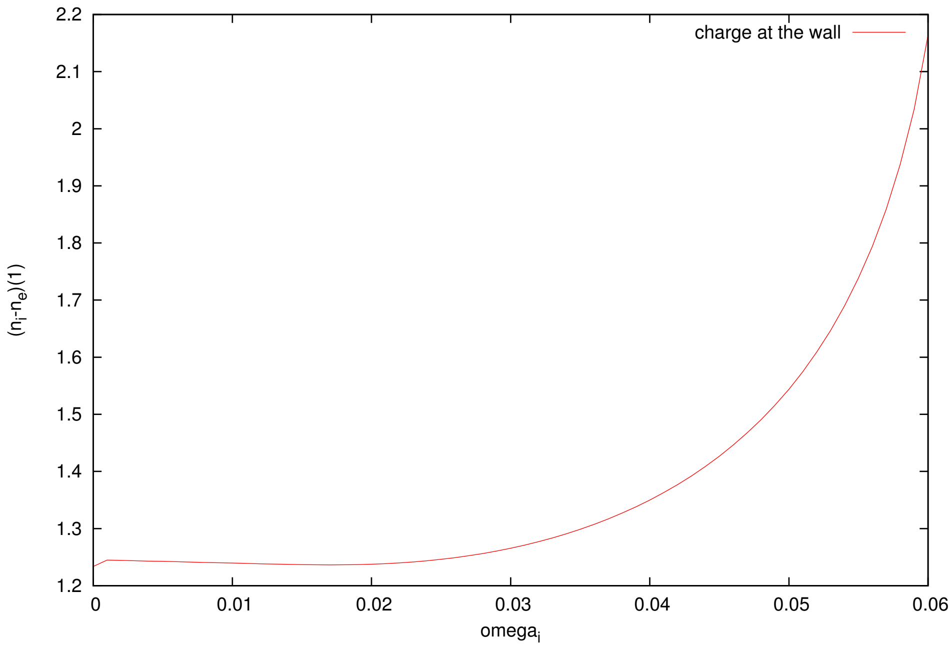


Figure 4.5 – Graph of the function $\omega_i \mapsto (n_i - n_e)(1)$ where ϕ_w is the solution to (4.47) for the first test case.

Second set test case:

For this second test case, the boundary condition is of the form

$$f_i^{in}(v_x, v_y, v_z) = \frac{1}{(2\pi)^{\frac{3}{2}}\sigma^3} v_y^2 e^{-\frac{(v_x^2+v_y^2+v_z^2)}{2\sigma^2}} \tag{4.77}$$

for $(v_x, v_y, v_z) \in (0 + \infty) \times \mathbb{R} \times (-\infty, -\frac{\omega_i}{2})$ with $\sigma = 1.5$. It is inspired from the work of Manfredi and Coulette in [43] where the authors consider a boundary condition aligned with the magnetic field. Our model is different from the one considered in [43] since we assume no particles with v_z greater than $-\frac{\omega_i}{2}$ which induces an anisotropy in the distribution function. The results presented for the second test case are qualitatively the same as the first test case. The bounds on the value of the magnetic field is approximately $\omega_i^c(f_i^{in}) \approx 0.063$. In table 4.2, we summarize the different parameters considered for the numerical simulations.

ω_i	ε	$\frac{\omega_{ci}}{\omega_{pi}}$	r_i^{in}
$\omega_i^1 = 0$	0.5	0	$+\infty$
$\omega_i^2 = 0.03$	0.5	$2.0E^{-2}$	63.3
$\omega_i^3 = 0.06$	0.5	$5.0E^{-2}$	31.7

Table 4.2 – Values of the different parameters for the second boundary condition f_i^{in} given by (4.77).

In figure 4.8, we represent the value of the wall potential ϕ_w solution to (4.47). In figure 4.9, we

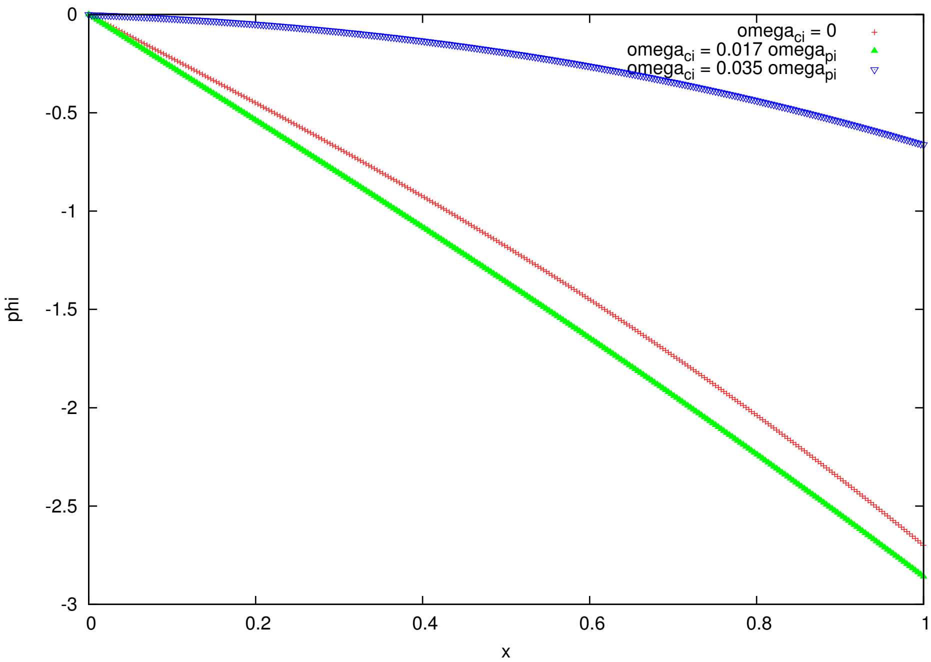


Figure 4.6 – Graph of the function $x \in [0,1] \mapsto \phi_\varepsilon^h(x)$ where ϕ_ε^h for $\varepsilon = \varepsilon^c(\omega_i^3) \approx 0.8$ (for the first test case) with different ratio $\frac{\omega_{ci}}{\omega_{pi}}$ as indicated.

represent the charge at the wall $n_i(1) - n_e(1) = \rho_i(1, \phi_w) - \rho_e(1, \phi_w)$ as functions of ω_i. The definition domain of these functions is the interval $[0, \omega_i^c]$ where $\omega_i^c(f_i^{in}) > 0$ is such that the inequality (4.49) is an equality. Its value is approximately 0.063. We observe that the wall potential seems to be a non decreasing function. The charge at the wall seems positive and non decreasing. As the intensity of the magnetic field increases the wall potential in absolute value decreases and the charge at the wall increases.

In figure 4.10 and 4.11, we represent the electrostatic potential and the charge density for the three different values of the magnetic field : $w_i^1 = 0$, $\omega_i^2 = 0.03$ and $\omega_i^3 = 0.06$. The corresponding critical value of the normalized Debye length for ω_i^3 is approximately $\varepsilon^c(\omega_i^3) \approx 0.5$. We see on figure 4.11 that increasing the magnetic field results in the charge density being everywhere increased except at $x = 0$ where the neutrality holds. The neutrality breaks when $x > 0$. For the two lowest value of the magnetic field we have considered, the charge seems non positive near $x = 0$ and then positive. For the highest value of the magnetic field, the charge is non negative everywhere.

Trajectories of the particles

In the following illustrations, we have considered the second boundary condition and we have taken $\omega_i = \omega_i^3 = 0.06$ which yields $\varepsilon^c(\omega_i^3) \approx 0.5$. Then the discrete potential was computed for $\varepsilon = 0.5$, this yields the ratio $\frac{\omega_{ci}}{\omega_{pi}} = 0.05$ and $r_i^{in} = 31.7$. We give an illustration of three electrons trajectories and three ions trajectories. These all start at $t = 0$ at the point $(x = 0, y = 0, z = 0)$ but with different initial velocities. As predicted in the study of the linear Vlasov system, an electron that enters at $x = 0$ with $v_z \geqslant E(\phi_w)$ will reach the wall, while an electron with $v_z < E(\phi_w)$ returns at $x = 0$ or reach the wall at $x = 1$ depending on its v_x velocity. As for the ions, whenever they enter at $x = 0$ with $v_z < -\frac{\omega_i}{2}$ they necessarily reach the wall $(x = 1)$. We also observe from the difference in scale of the

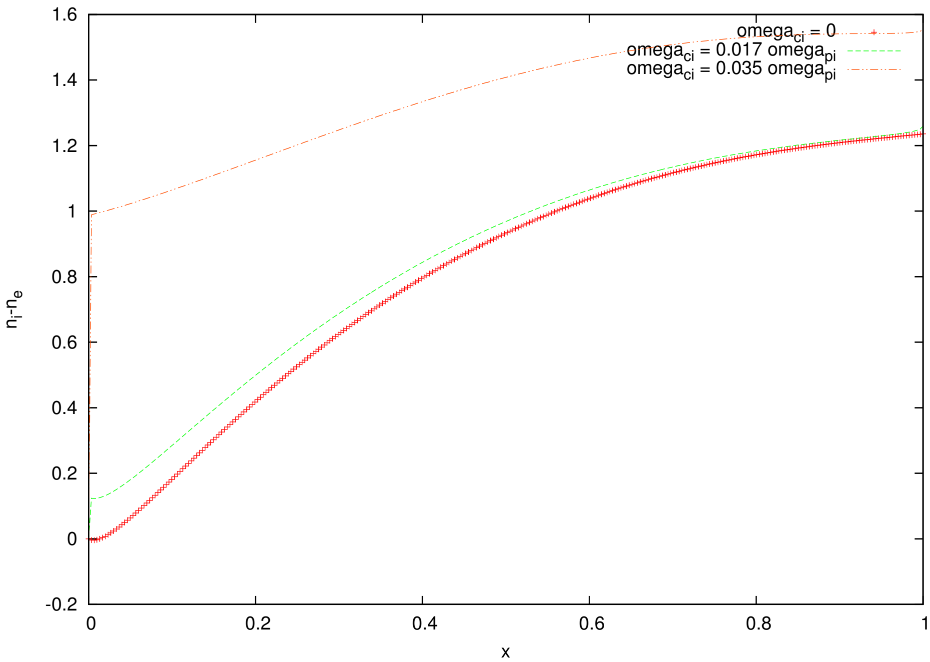


Figure 4.7 – Graph of the function $x \in [0,1] \mapsto (n_i - n_e)(x)$ and with different ratio $\frac{\omega_{ci}}{\omega_{pi}}$ (for the first test case) as indicated.

y-axis, the different time scales between ions and electrons. Indeed, because ions and electrons have the same constant velocity $v_y^0 = 0.01$ a quick computation shows that the approximate time for an electron to reach $x = 1$ is $t_e^{out} \approx 0.002s$ while for the ions it is $t_i^{out} \approx 1.5s$. The ratio $\frac{t_e^{out}}{t_i^{out}}$ is of the order of the mass ratio $\mu = \frac{1}{3672}$.

4.6.4 Interpretation of the results and extension of the bounds on ω_i and ε.

For the parameters corresponding to the applicability domain, the ion Larmor radius r_i^{in} was way larger than the normalized Debye length ε. For such physical parameters, we see that increasing the intensity of the magnetic field results in an increasing charge density close the wall. We also observe theoretically from the inequality (4.49) that it tends to limit the flux of ions perpendicular to the wall. Both these behaviors are in agreement with the numerical findings in [43]. We also observe that the potential drop decreases. Concerning the extension of the domain of applicability for a given ion boundary condition f_i^{in}, we believe that it is possible to extend the model to positive wall potential. We have not considered this situation here but the work of Moritz and al [48] does. Especially, they observed a negative charge separation at the wall that seems to increase with increasing magnetic field intensity and a relative wall potential that is positive. Their results are not in contradiction with ours since in their work, the authors consider magnetic field intensities that seem to be always greater than the values we consider in this work. It thus questions the extension of our model to greater value of the magnetic field. We recall that in the model we consider, the potential was assumed to be non positive. Since the presence of a significant intense magnetic field generates positive potential at the wall, it is natural that our model is only applicable to low magnetic fields. As far as the normalized Debye length is considered, we observe

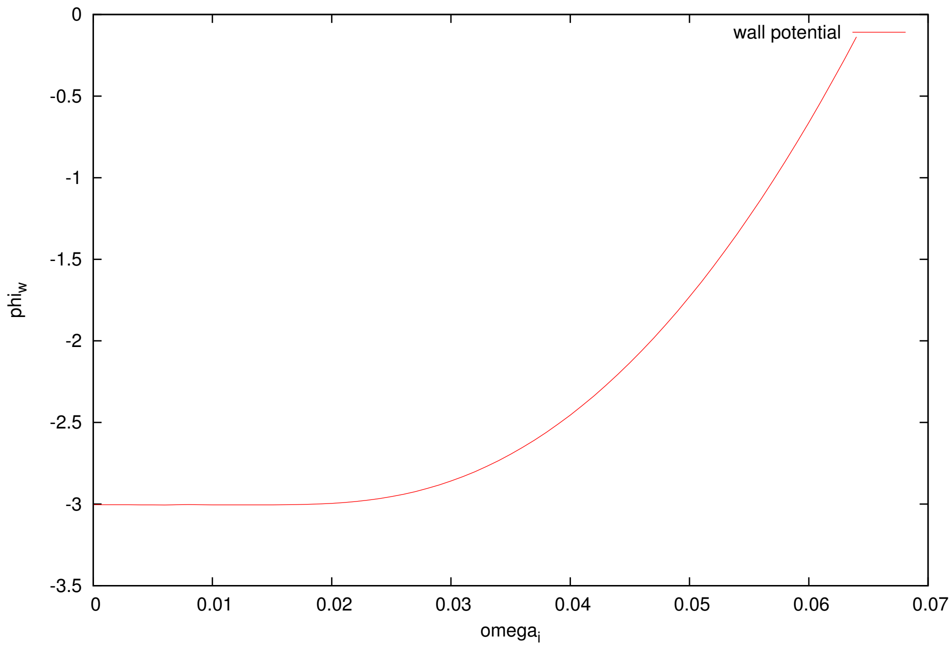


Figure 4.8 – Graph of the function $\omega_i \mapsto \phi_w$ where ϕ_w is the solution to (4.47) for the second test case. Here the critical value ω_i^c is slightly larger than 0.063 for which $\phi_w = 0$.

that increasing ω_i results in a critical value $\varepsilon^c(\omega_i)$ (that is a lower bound on ε) being increased. It therefore seems that these two parameters are related to one another. The physical interpretation is not obvious a priori, however what we observed at the numerical level, is that for fixed value of the magnetic field intensity $\omega_i > 0$, when the normalized Debye length ε becomes to small, the electrostatic potential ϕ_ε^h computed via our projected gradient algorithm saturates the constraint that $\phi_\varepsilon^h \leqslant 0$ and is no longer an acceptable numerical solution to (NLP-MMAG). We therefore conjecture that for small normalized Debye length, there might be a transition region between the plasma and the wall where the potential ϕ_ε^h becomes positive and oscillates. Since in our model we constrain the electrostatic potential ϕ_ε^h to be non negative everywhere it does not allow to capture a solution for small value of the normalized Debye length. The extension of the model for smaller value of the normalized Debye length, or higher value of the magnetic field, is an open question.

4.7 Conclusion

We have proposed and studied a stationary and one dimensional (with 3 dimensional velocities) plasma-wall interaction model, based on a bi-kinetic description of ions and electrons. We have shown how to reduce the study of the full Vlasov-Poisson system (4.14)-(4.21) to the study of a non linear Poisson equation. In this model incoming electrons were considered Maxwellian. As for the ions, our model supports a large class of incoming velocity distributions f_i^{in} and we have identified three conditions for f_i^{in} so that the wall potential is well-defined. The first condition is a lower bound on the derivative of the electric field, the second one is a support condition on the distribution f_i^{in} that conveys the idea that no particles enters with v_z velocities greater than some bound that only depends the intensity of the

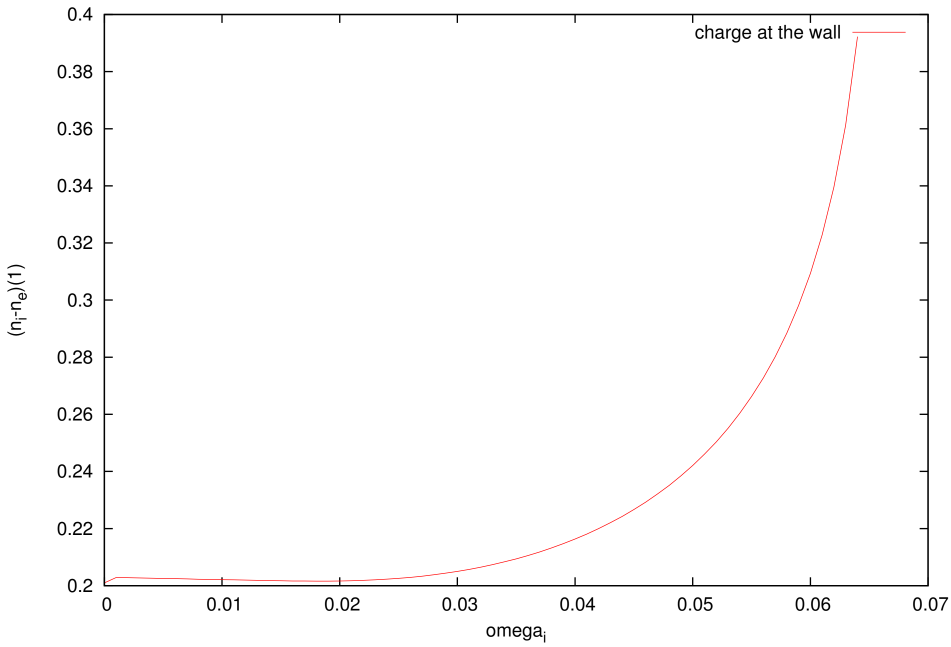


Figure 4.9 – Graph of the function $\omega_i \mapsto (n_i - n_e)(1)$ where ϕ_w is the solution to (4.47) for the second test case.

magnetic field. This condition also turns out to prevent ions particles from being confined by the magnetic field. The third condition takes the form of an upper bound on the incoming ion flux perpendicular to the wall. Then we have proposed to study the non linear Poisson equation as a minimization problem. This approach provides us with the existence of minimizers and a quantitative estimate for the electric field. It nevertheless turns out to be insufficient to ensure the minimizers to be critical points, and thus leaves the question of existence and uniqueness for the non linear Poisson equation unresolved. The identification of boundary conditions f_i^{in} that provides solutions to the non linear Poisson equations is an open problem. Results of two sets of numerical simulations were finally presented for a range of parameters that seems to provide solutions for the non linear Poisson equation. Especially for each ion incoming distribution f_i^{in} we considered, we have identified some numerical bounds on the parameters ω_i and ε outside which it is no longer possible to compute a non positive wall potential and a numerical potential that solves the non Poisson problem. Inside these bounds, it is possible to do so. The first set was concerned with the perturbation of a physically based sheath problem while the second one was inspired from [43]. For the range of parameters we have considered, results show that increasing the intensity of the magnetic field tends to limit the ion flux perpendicular to the wall which results in a wall potential that diminishes and a charge separation at the wall that increases. These results are qualitatively in good agreement with the results obtained by Manfredi and Coulette [43] for a magnetic field with a low incidence angle. The results we have presented are made in a regime where the ion Larmor radius is way larger than the Debye length. Other regime were not considered. Besides, in our model we have made an assumption on the ion boundary condition f_i^{in} that prevents the ions to be confined by the magnetic field. This assumption clearly implies that incoming ions are fated to reach the wall (even with a weak electric-field). The extension of this work to physical parameters outside the bounds we have already determined is an open and delicate problem. In particular, the work of Moritz and al [48] seems to show off non trivial physical

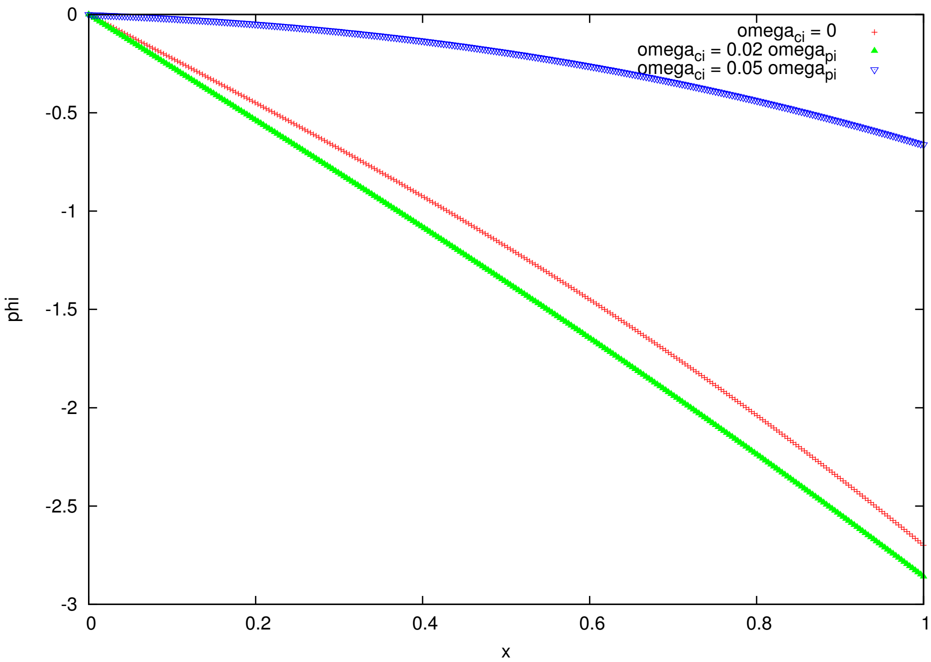


Figure 4.10 – Graph of the function $x \in [0,1] \mapsto \phi^h_\varepsilon(x)$ where ϕ^h_ε for $\varepsilon = 0.5$ (for the second test case) with different ratio $\frac{\omega_{ci}}{\omega_{pi}}$ as indicated.

solutions in the case of a strongly magnetized plasma. Especially, they observed a negative charge at the wall. Especially, one could think to relax the assumption on the support of f_i^{in} that induces an anisotropy in the distribution function, that seems not to be the practical situation. We hope this work to be the first one of a series of future works where less restrictives assumptions will be made.

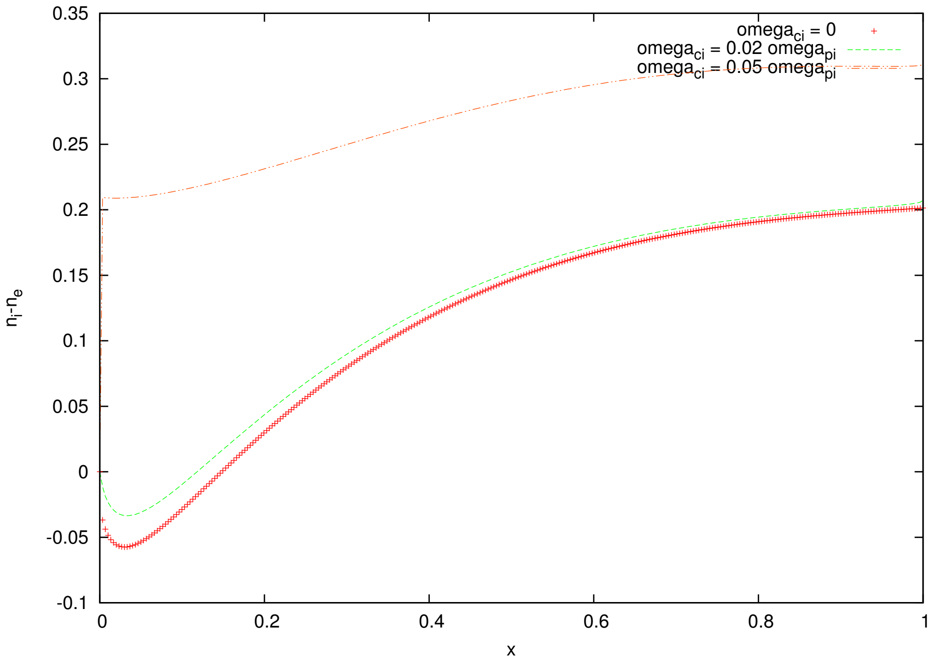


Figure 4.11 – Graph of the function $x \in [0, 1] \mapsto (n_i - n_e)(x)$ with different ratio $\frac{\omega_{ci}}{\omega_{pi}}$ (for the second test case) as indicated.

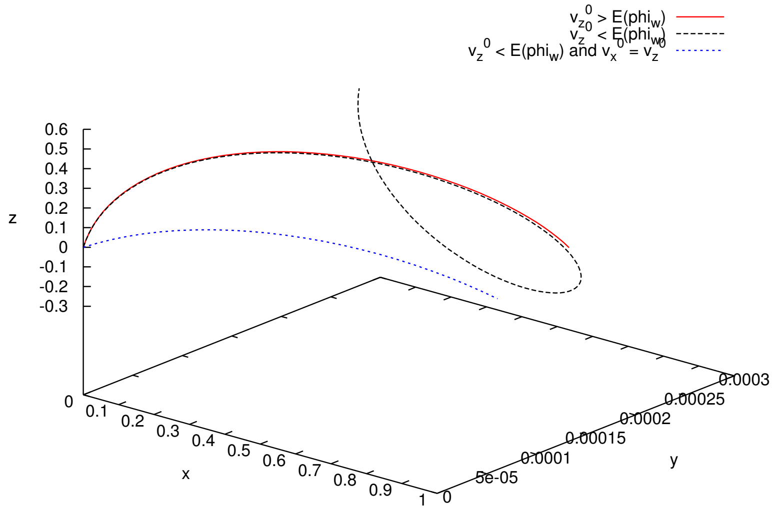


Figure 4.12 – Three electrons trajectories starting from the point $(0, 0, 0)$ with different initial velocities. The red curve corresponds to an initial velocity $v_x^0 = 0.01$, $v_y = 0.01$ and $v_z^0 = E(\phi_w) + 1$. The black curve corresponds to to an initial velocity $v_x^0 = 0.01$, $v_y = 0.01$ and $v_z^0 = E(\phi_w) - 1$. The blue curve corresponds to to an initial velocity $v_x^0 = v_z^0$ with $v_z^0 = E(\phi_w) - 1$ and $v_y^0 = 0.01$.

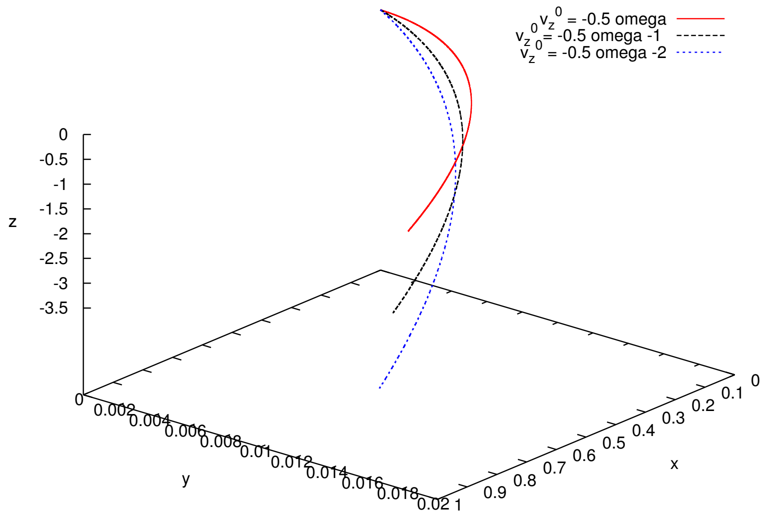


Figure 4.13 – Three ions trajectories starting from the point $(0, 0, 0)$ with different initial velocities. The red curve corresponds to an initial velocity $v_x^0 = 0.01$, $v_y = 0.01$ and $v_z^0 = -\frac{\omega_i}{2}$. The black curve corresponds to to an initial velocity $v_x^0 = 0.01$, $v_y = 0.01$ and $v_z^0 = -\frac{\omega_i}{2} - 1$. The blue curve corresponds to to an initial velocity $v_x^0 = 0.01$, $v_y = 0.01$ and $v_z^0 = -\frac{\omega_i}{2} - 2$.

Conclusion and perspectives

In this thesis we have proposed to study three different models of plasma-wall interaction. These models were based on a kinetic description of the ions and electrons through some Vlasov-Poisson or Vlasov-Ampère systems with boundary conditions. Ions and electrons were emitted at one side, while at the other side ions were absorbed and electrons partially re-emitted. In these models, in addition to the classical unknowns that are the electrostatic potential and the distribution functions of each species, we have also considered the boundary conditions as unknown parameters that needed to be determined so that the stationary solutions to these Vlasov-Poisson systems satisfy two conditions:

- In the perpendicular direction to the wall, the flux of ions equals the flux of electrons at the wall. It is also called the ambipolarity.
- In the core plasma, the charge takes an a priori given value. It could be a null charge or a charge imbalance.

For the first and second model, the magnetic-field was neglected, while in the third one it was considered. For the first model, we have proven, provided the incoming ions velocity distributions satisfies a kinetic variant of the Bohm criterion and an upper bound on the mean velocity of the incoming ions, that the model was well-posed for any value of the Debye length and for a large range of re-emission parameter for the electrons. More precisely, we have proven that the wall potential was the unique solution of a non linear equation, and that the electrostatic potential was the unique solution of a non linear Poisson equation. Besides, we have shown that the electric-field is non negative and the charge is non non negative everywhere. We have also given some quantitative bounds on the charge and the electric field that scaled respectively as the Debye length and as is its inverse. Numerical simulations have shown that when the Debye length becomes small, a thin positively charged layer of several Debye length develops near the wall. The second model was a non-stationary Vlasov-Ampère model for which the solutions of the first model are equilibrium. We have proven that when considering the ions as fixed, the equilibrium is linearly stable. The last model was dedicated to the extension of the first model. In particular, we added a constant magnetic field. We have proven, that provided the incoming ions satisfies a moment conditions that takes the form of an upper-bound on their mean velocity, the wall potential is the unique solution of a non linear equation and that the electrostatic potential solves a non linear Poisson problem. We have given a minimization formulation for the non linear Poisson problem and have proven the existence of a minimizer. Numerical simulations have revealed some numerical bounds on the value of the magnetic field and on the normalized Debye length. Inside these bounds, the numerical methods we have implemented are able to capture a good approximation of the solution to the non linear Poisson problem. We have observed that increasing the value of the magnetic field results in a charge that is increased and a wall potential (in absolute value) that diminishes. We have also illustrated the predicted behavior of the ions and electrons by plotting some trajectories. Incoming ions were fated to reach the wall. We mention that one of the key ingredient for the first and the third model was the construction of enough invariants to the characteristics.

The perspectives of these works are wide open. For the first model, one could consider the study of an extended model where collision operators are included [60]. Also, one could study the same model but in a different geometry where the wall is no longer assimilated to a plane, but to a bent surface. It has practical applications, the well-known Langmuir probe problem [40] is an example and the modeling of plasma thrusters [22] is another example. We also mention that the first model, easily extends to the case where the incoming ion boundary condition is mono kinetic, that is a Dirac distribution. As far as the second model is concerned, the perspective is straightforward. One could consider the study of the stability when ions are no longer fixed. It would be possible that the system is in fact unstable [10]. Numerical investigations are a first step towards an answer. Some numerical experiments carried by Güclu in [29] seem the show off the stability of the ions. The third model contains a lot of open questions: well-posedness of the non linear Poisson problem, extension of the work to more general ion boundary conditions. Let us also mention one of the difficulty we encountered in this work. This difficulty comes from the minimization technique we employed to solve the non linear Poisson problem. More precisely, because we minimize a certain function J_ε on a closed convex set that is not the whole space, the minimizer is not necessarily a critical point. We recall here the following proposition.

Proposition 4.7.1. *Let $\alpha \in [0,1)$,$\omega_i > 0$, $\mu > 0$, $f_i^{in} \in \mathcal{I}_{ad}(\alpha,\omega_i,\mu)$ and $\varepsilon > 0$. Let $\phi := \phi_\varepsilon \in V_{ad}(\alpha,\omega_i,\mu)$ a minimizer of J_ε. Then the following variational inequality holds*

$$dJ_\varepsilon(\phi)(h) \geqslant 0 \qquad \textit{pour tout } h \in V_0 \textit{ such that } \phi + h \in V_{ad}(\alpha,\omega_i,\mu). \tag{4.78}$$

Moreover, $\phi \in W^{2,\infty}(0,1) \cap C^1[0,1]$ and

$$-\varepsilon^2\phi''(x) = \frac{\partial}{\partial t}\mathcal{U}(x,\phi(x)) \qquad a.e \textit{ in } \mathcal{O} := \{x \in (0,1) \mid \underline{q}(x) < \phi(x) < 0\}, \tag{4.79}$$

$$-\varepsilon^2\phi''(x) \leqslant \frac{\partial}{\partial t}\mathcal{U}(x,\phi(x)) \qquad a.e \textit{ in } \mathcal{F}_1 := \{x \in (0,1) \mid \phi(x) = 0\}, \tag{4.80}$$

$$-\varepsilon^2\phi''(x) \geqslant \frac{\partial}{\partial t}\mathcal{U}(x,\phi(x)) \qquad a.e \textit{ in } \mathcal{F}_2 := \{x \in (0,1) \mid \phi(x) = \underline{q}(x)\}. \tag{4.81}$$

In particular, when the minimizer vanishes in $\mathcal{F}_1$, we get the inequality $\rho_i(x,0) - \rho_e(x,0) \geqslant 0$. An idea was to determine f_i^{in} such that the inequality is an equality. For the first model, we stumbled upon a similar difficulty but we have managed to overcome it. Here the question stays open. Since $\rho_i(0,0) - \rho_e(0,0) = 0$, a clue was to make constant $\rho_i(x,0) - \rho_e(x,0)$ on an interval included in $\mathcal{F}_1$ and containing $x = 0$. We have been able to prove that it is not possible to make $\rho_i(x,0)$ and $\rho_e(x,0)$ constant independently to one another in a neighborhood of $x = 0$, but the original question is still open.

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Étude mathématiques et simulations numériques de modèles de gaines bi-cinétiques.

Abstract

Les résultats présentés dans cette thèse portent sur la construction et la simulation numérique de modèles théoriques de plasmas en présence d'une paroi absorbante. Ces modèles se basent sur des systèmes de Vlasov-Poisson ou Vlasov-Ampère à deux espèces en présence de conditions limites. Les solutions stationnaires recherchées vérifient l'équilibre des flux de charges dans la direction perpendiculaire à la paroi. Cette propriété s'appelle l'ambipolarité. A travers l'étude d'une équation de Poisson non linéaire, on montre le caractère bien posé d'un système de Vlasov-Poisson stationnaire 1d-1v pour lequel on détermine des distributions de particules entrantes et un potential au mur qui induisent l'ambipolarité et une densité de charge positive. On donne également une estimation de la taille de la couche limite au mur. Ces résultats sont illustrés numériquement. On prouve ensuite la stabilité linéaire des solutions stationnaires électroniques pour un modèle de Vlasov-Ampère instationnaire. Enfin, on étudie un modèle de Vlasov-Poisson stationnaire 1d-3v en présence d'un champ magnétique constant et parallèle à la paroi. On détermine les distributions de particules entrantes et un potentiel au mur qui induisent l'ambipolarité. On étudie une équation de Poisson non linéaire associée au modèle à l'aide d'une fonctionnelle non linéaire d'énergie qui admet des minimiseurs. On établit des bornes de paramètres à l'intérieur desquelles notre modèle s'applique et on propose une interprétation des résultats.

Keywords: système de vlasov-poisson, système de vlasov-ampère, equation de poisson non linéaire, gaine de debye, champ magnétique parallèle, critère de bohm, potential flottant

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Résumé

This thesis focuses on the construction and the numerical simulation theoretical models of plasmas in interaction with an absorbing wall. These models are based on two species Vlasov-Poisson or Vlasov-Ampère systems in the presence of boundary conditions. The expected stationary solutions must verify the balance of the flux of charges in the orthogonal direction to the wall. This feature is called the ambipolarity. Through the study of a non linear Poisson equation, we prove the well-posedness of 1d-1v stationary Vlasov-Poisson system, for which we determine incoming particles distributions and a wall potential that induces the ambipolarity as well as a non negative charge density hold. We also give a quantitative estimates of the thickness of the boundary layer that develops at the wall. These results are illustrated numerically. We prove the linear stability of the electronic stationary solution for a non-stationary Vlasov-Ampère system. Finally, we study a 1d-3v stationary Vlasov-Poisson system in the presence of a constant and parallel to the wall magnetic field . We determine incoming particles distributions and a wall potential so that the ambipolarity holds. We study a non linear Poisson equation through a non linear functional energy that admits minimizers. We established some bounds on the numerical parameters inside which, our model is relevant and we propose an interpretation of the results.

Mots clés : vlasov-poisson system, vlasov-ampère system, non linear poisson equation, debye sheath, parallel magnetic field, bohm criterion, floating potential

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